

Curvature of Almost Quaternion-Hermitian Manifolds

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Abstract. We study the decomposition of the Riemannian curvature R tensor of an almost quaternion-Hermitian manifold under the action of its structure group $Sp(n)Sp(1)$. Using the minimal connection, we show that most components are determined by the intrinsic torsion ξ and its covariant derivative $\tilde{\nabla}\xi$ and determine relations between the decompositions of $\xi \otimes \xi$, $\tilde{\nabla}\xi$ and R . We pay particular attention to the behaviour of the Ricci curvature and the q-Ricci curvature.

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Contents

1	Introduction	1
2	Preliminaries	3
3	Quaternionic decomposition of curvature	8
4	Almost quaternion-Hermitian decomposition of curvature	12
5	Intrinsic torsion	18
6	Curvature and intrinsic torsion	22

1 Introduction

An object of fundamental importance in Riemannian geometry is the curvature tensor R . As a $(0, 4)$ -tensor, R satisfies a number of algebraic symmetry conditions, including the Bianchi identity. The presence of an additional geometric structure on the manifold gives rise to a decomposition of the

curvature in to components, each satisfying additional symmetry relations. Further information on the geometric structure may then imply the vanishing of some of these components. The most celebrated examples of this come from the holonomy classification of Berger [1] where nearly all the possible non-trivial reductions of the Riemannian holonomy of an irreducible structure give solutions of the Einstein equations, see [3].

Tricerri & Vanhecke [19] were the first to make a detailed study of the general decompositions for one particular class of geometric structures, namely almost Hermitian structures, i.e., manifolds with a metric and compatible almost complex structure. The purpose of this paper is to extend these techniques to almost quaternion-Hermitian manifolds. These are manifolds with a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and local triples of almost complex structures I, J, K satisfying the quaternion identities.

For the almost Hermitian case, the geometry is determined by a $U(n)$ -structure. For almost quaternion-Hermitian manifolds the structure group is $Sp(n)Sp(1)$. Both groups appear on Berger's list of Riemannian holonomy groups and are of fundamental importance in the study of non-linear supersymmetric σ -models in physics.

The first step in the study of curvature on these manifolds is to decompose the space \mathcal{R} of $(0, 4)$ -curvature tensors under the action of the structure group. It is helpful to do this in two steps. The space \mathcal{R} is invariant under the larger group $GL(n, \mathbb{H})Sp(1)$ that preserves the space of compatible almost complex structures but not the metric. We thus first decompose \mathcal{R} under the action of $GL(n, \mathbb{H})Sp(1)$, see §3, and then determine the refinement of this decomposition under the action of the smaller group $Sp(n)Sp(1)$, see §4. This mirrors the approach of Falcitelli et al. [7] in the almost Hermitian case, see also [11]. Note that although $GL(n, \mathbb{H})Sp(1)$ is the structure group of an almost quaternionic structure, the metric has been used to convert the curvature tensor from type $(1, 3)$ to type $(0, 4)$ by lowering an index, and so the first step does not correspond to decomposition results for curvature of almost quaternionic manifolds.

Given a Riemannian G -structure, there is a distinguished compatible connection $\tilde{\nabla}$, the minimal connection, characterised by having the smallest torsion pointwise. The torsion of $\tilde{\nabla}$ is called the intrinsic torsion of the G -structure is determined by the tensor $\xi = \tilde{\nabla} - \nabla$, where ∇ is the Levi-Civita connection of the metric. The vanishing of ξ is equivalent to the reduction of the holonomy to G . In general, the intrinsic torsion ξ splits up in to a number of components under the action of G . Vanishing of certain components often

correspond to interesting geometric properties, and structures with specific torsion are of increasing importance in theoretical physics.

For $G = Sp(n) Sp(1)$ the intrinsic torsion ξ splits in to six components [16]. These contribute to the curvature tensor R in two different ways; via components of $\xi \otimes \xi$ and via components $\tilde{\nabla}\xi$. This information determines directly all the components of R transverse to the space \mathcal{QK} of curvature tensors of quaternionic Kähler manifolds, i.e., manifolds where the holonomy reduces to $Sp(n) Sp(1)$. Since $\dim \mathcal{R} = \frac{4}{3}n^2(16n^2 - 1)$ and $\dim \mathcal{QK} = \frac{1}{6}(4n^4 + 12n^3 + 11n^2 + 3n + 6)$, this means that nearly all components of R are determined by ξ and its derivative.

In §6, we compute the contribution of ξ to the components of R and display the results in tables. Particular attention is paid to the contributions to the Ricci curvature Ric and its quaternionic partner the q-Ricci curvature Ric^q . For these two tensors, it is only the scalar parts that remain undetermined and we show how even these may be found by invoking additional information from the curvature of the bundle of compatible local almost complex structures. The paper closes with a number of consequences for particular types of almost quaternion-Hermitian manifolds. For examples of such structures we refer the interested reader to Cabrera & Swann [12], where it is also shown how the components of ξ may be efficiently computed via the exterior algebra.

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2 Preliminaries

Let \mathcal{V} be an m -dimensional real vector space. The space of Riemannian curvature tensors \mathcal{R} on \mathcal{V} consists of those tensors R of type $(0, 4)$ which satisfies the same symmetries as the Riemannian curvature tensor of a Riemannian manifold. This is summarised by saying that \mathcal{R} is the kernel of the mapping

$$S^2(\Lambda^2 \mathcal{V}^*) \rightarrow \Lambda^4 \mathcal{V}^*, \quad (2.1)$$

defined by wedging two-forms together.

When there is a positive definite inner product $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle$, defined on \mathcal{V} , then \mathcal{V} is a representation of the orthogonal group $O(m)$ and we can

consider the map $\text{Ric}: \mathcal{R} \rightarrow S^2\mathcal{V}^*$, given by $\text{Ric}(R)(x, y) = R(x, e_i, y, e_i)$, where $\{e_1, \dots, e_m\}$ is an orthonormal basis for vectors and we use the summation convention. This notation and such a convention will be used in the sequel. The map Ric is $O(m)$ -invariant and $\text{Ric}(R)$ is called the *Ricci tensor* associated to R . The *scalar curvature* $\text{scal}(R)$ of R is the trace of $\text{Ric}(R)$.

There is a natural extension of the inner product $\langle \cdot, \cdot \rangle$ to the space of p -forms $\Lambda^p\mathcal{V}^*$ defined by

$$\langle a, b \rangle = \frac{1}{p!} a(e_{i_1}, \dots, e_{i_p}) b(e_{i_1}, \dots, e_{i_p}).$$

On the other hand, we will also consider \mathcal{V} equipped with three almost complex structures I, J and K satisfying the same identities as the imaginary units of quaternion numbers, i.e., $I^2 = J^2 = -1$ and $K = IJ = -JI$. In such a case $m = 4n$, and \mathcal{V} is a representation of the subgroup $GL(n, \mathbb{H}) Sp(1)$ of $GL(4n, \mathbb{R})$ characterised by the fact that it preserves the three-dimensional vector space \mathcal{G} of endomorphisms of \mathcal{V} generated by I, J and K . A triple I', J' and K' is said to be an *adapted basis* for \mathcal{G} , if they generate \mathcal{G} , satisfy the same identities as the unit imaginary quaternions. One finds that for $A = I', J', K'$ we have $A = a_A I + b_A J + c_A K$ with $a_A^2 + b_A^2 + c_A^2 = 1$.

Furthermore, we will also consider both situations simultaneously: \mathcal{V} equipped with an inner product $\langle \cdot, \cdot \rangle$ and three almost complex structures I, J and K satisfying the above mentioned quaternionic identities and the compatibility condition with the inner product, $\langle Ax, Ay \rangle = \langle x, y \rangle$, for $A = I, J, K$. In this case, we have the three Kähler forms given by $\omega_A(x, y) = \langle x, Ay \rangle$, $A = I, J, K$, and the four-form Ω defined by

$$\Omega = \sum_{A=I,J,K} \omega_A \wedge \omega_A. \quad (2.2)$$

The $4n$ -form Ω^n can be used to fix an orientation and \mathcal{V} is a representation of the subgroup $Sp(n) Sp(1)$ of $SO(4n)$ consisting by those elements which preserve \mathcal{G} . Alternatively, $Sp(n) Sp(1)$ can be defined as consisting of those elements of $O(4n)$ which preserve Ω .

The following notation will be used in this paper. If b is a $(0, s)$ -tensor, we write

$$\begin{aligned} A_{(i)} b(X_1, \dots, X_i, \dots, X_s) &= -b(X_1, \dots, AX_i, \dots, X_s), \\ Ab(X_1, \dots, X_s) &= (-1)^s b(AX_1, \dots, AX_s), \end{aligned}$$

for $A = I, J, K$.

There is an $Sp(n) Sp(1)$ -invariant map $\text{Ric}^q: \mathcal{R} \rightarrow \otimes^2 \mathcal{V}^*$ given by

$$\text{Ric}^q(R)(x, y) = \sum_{A=I,J,K} R(x, e_i, Ay, Ae_i).$$

The tensor $\text{Ric}^q(R)$ will be called the *q-Ricci tensor* and the trace $\text{scal}^q(R)$ of $\text{Ric}^q(R)$ will be referred as the *q-scalar curvature* of R . If we write

$$\text{Ric}_A^* = R(x, e_i, Ay, Ae_i),$$

it is not hard to prove that

$$\sum_{A=I,J,K} A(\text{Ric}_B^*)_{\mathbf{a}} = -(\text{Ric}_B^*)_{\mathbf{a}}, \quad \langle \text{Ric}_B^*, \omega_B \rangle = 0,$$

where $(\text{Ric}_B^*)_{\mathbf{a}}$ is the skew-symmetric part of Ric_B^* . Moreover, for all cyclic permutations of I, J, K , we have

$$\langle \text{Ric}_I^*, \omega_J \rangle = -\langle \text{Ric}_K^*, \omega_J \rangle.$$

Hence the skew-symmetric part $\text{Ric}_{\mathbf{a}}^q$ of the tensor Ric^q satisfies the conditions:

- (i) $\sum_{A=I,J,K} A \text{Ric}_{\mathbf{a}}^q = -\text{Ric}_{\mathbf{a}}^q$, and
- (ii) $\langle \text{Ric}_{\mathbf{a}}^q, \omega_A \rangle = 0$, for $A = I, J, K$,

which characterises the irreducible $Sp(n) Sp(1)$ -module $\Lambda_0^2 ES^2 H \subset \Lambda^2 \mathcal{V}^*$ of skew-symmetric two-forms. Thus $\text{Ric}_{\mathbf{a}}^q \in \Lambda_0^2 ES^2 H$. In summary, we have

$$\begin{aligned} \text{Ric}^q &\in S^2 \mathcal{V}^* + \Lambda_0^2 ES^2 H = \mathbb{R}g + \Lambda_0^2 E + S^2 ES^2 H + \Lambda_0^2 ES^2 H \quad \text{and} \\ \text{Ric} &\in S^2 \mathcal{V}^* = \mathbb{R}g + \Lambda_0^2 E + S^2 ES^2 H, \end{aligned}$$

where $\Lambda_0^2 E$ consists of trace-free symmetric two-forms b such that $Ab = b$, $A = I, J, K$, and $S^2 ES^2 H$ consists of those such that $\sum_A Ab = -b$.

Now, we recall some facts about quaternionic structures in relation with representation theory. We will follow the *E-H*-formalism used in [14, 16, 17] and we refer to [5] for general information on representation theory. Thus, E is the fundamental representation of $GL(n, \mathbb{H})$ on $\mathbb{C}^{2n} \cong \mathbb{H}^n$ via left multiplication by quaternionic matrices, considered in $GL(2n, \mathbb{C})$, and H is the

representation of $Sp(1)$ on $\mathbb{C}^2 \cong \mathbb{H}$ given by $q \cdot \zeta = \zeta \bar{q}$, for $q \in Sp(1)$ and $\zeta \in \mathbb{H}$.

An irreducible representation of $GL(n, \mathbb{H})$ is determined by its dominant weight $(\lambda_1, \dots, \lambda_n)$, where λ_i are integers with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. This representation will be denoted by $U^{\lambda_1, \dots, \lambda_r}$, where r is the largest integer such that $\lambda_r > 0$. Likewise, $U^{*\lambda_1, \dots, \lambda_r}$ will denote the dual representation of $U^{\lambda_1, \dots, \lambda_r}$. Familiar notation is used for some of these modules, when possible. For instance, $U^k = S^k E$, the k th symmetric power of E , and $U^{1, \dots, 1} = \Lambda^r E$, where there are r ones in exponent. Likewise, $S^k E^*$ and $\Lambda^r E^*$ will be the respective dual representations.

On E , there is an invariant complex symplectic form ω_E and a Hermitian inner product given by $\langle x, y \rangle_{\mathbb{C}} = \omega_E(x, \tilde{y})$, for all $x, y \in E$, and being $\tilde{y} = jy$ ($y \mapsto \tilde{y}$ is a quaternionic structure map on $E = \mathbb{C}^{2n}$ considered as left complex vector space). The mapping $x \mapsto x^\omega = \omega_E(\cdot, x)$ gives us an identification of E with its dual E^* . In using group representations, this identification works only with groups which preserve ω_E . For instance, we can not use such an identification for $GL(n, \mathbb{H})$ -representations. If $\{e_1, \dots, e_n, \tilde{e}_1, \dots, \tilde{e}_n\}$ is a complex orthonormal basis for E , then $\omega_E = e_i^\omega \wedge \tilde{e}_i^\omega = e_i^\omega \tilde{e}_i^\omega - \tilde{e}_i^\omega e_i^\omega$, where we have omitted tensor product signs. The group $Sp(n)$ coincides with the subgroup of $U(2n)$ which preserves ω_E .

The $Sp(1)$ -module H will be also considered as left complex vector space. Regarding H as 4-dimensional real space with the Euclidean metric $\langle \cdot, \cdot \rangle$ such that $\{1, i, j, k\}$ is an orthonormal basis, the complex symplectic form ω_H is given by $\omega_H = (1^b \wedge j^b + k^b \wedge i^b) + i(1^b \wedge k^b + i^b \wedge j^b)$, where h^b is the real one-form given by $q \mapsto \langle h, q \rangle$. We also have the identification, $q \mapsto q^\omega = \omega_H(\cdot, q)$, of H with its dual H^* as complex space. On H , we have a quaternionic structure map given by $h = z_1 + z_2 j \mapsto \tilde{h} = jq = -\bar{z}_2 + \bar{z}_1 j$, where $z_1, z_2 \in \mathbb{C}$ and \bar{z}_1, \bar{z}_2 are their conjugates. On H^* , the structure map given by $h^\omega = -\tilde{h}^\omega$, this is based in the identities

$$\tilde{h}^\omega(q) = \overline{h^\omega(\tilde{q})} = \overline{\omega_H(\tilde{q}, h)} = -\omega_H(q, \tilde{h}) = -\tilde{h}^\omega(q).$$

From now on, we will take $h \in H$ such that $\langle h, h \rangle = 1$. Thus $\{h, \tilde{h}\}$ is a basis of the complex vector space H and $\omega_H = h^\omega \wedge \tilde{h}^\omega$. Finally, we point out that the irreducible representations of $Sp(1)$ are the symmetric powers $S^k H \cong \mathbb{C}^{k+1}$.

An irreducible representation of $Sp(n)$ is also determined by its dominant weight $(\lambda_1, \dots, \lambda_n)$, where λ_i are integers with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. This

representation will be denoted by $V^{\lambda_1, \dots, \lambda_r}$, where r is the largest integer such that $\lambda_r > 0$. Likewise, familiar notation is also used for some of these modules. For instance, $V^k = S^k E$, and $V^{1, \dots, 1} = \Lambda_0^r E$, where there are r ones in exponent and $\Lambda_0^r E$ is the $Sp(n)$ -invariant complement to $\omega_E \Lambda^{r-2} E$ in $\Lambda^r E$. Also K will denote the module V^{21} , which arises in the decomposition $E \otimes \Lambda_0^2 E \cong \Lambda_0^3 E + K + E$, where $+$ denotes direct sum.

Remark 2.1. Regarding complex and real representations: suppose V is a complex G -module equipped with a real structure, where G is a Lie group. Most of the time in this paper, V will also denote the real G -module which is the $(+1)$ -eigenspace of the structure map. The context should tell us which space are referring to. However, if there is risk of confusion or when we feel that a clearer exposition is needed, we will denote the second mentioned space by $[V]$.

Returning to our real vector space \mathcal{V} with the three almost complex structures I, J and K satisfying the quaternionic identities, we can consider \mathcal{V} as complex vector space by saying $(\lambda + i\mu)x = \lambda x + \mu Ix$, for all $\lambda + i\mu \in \mathbb{C}$ and $x \in \mathcal{V}$. Since $2n$ is the dimension of such a complex vector space, we will also write E when we are referring to \mathcal{V} as complex vector space. The dual vector space E^* of E consists of complex one-forms $a_{\mathbb{C}} = a + iIa$, for $a \in \mathcal{V}^*$, and has $ia_{\mathbb{C}} = -(Ia)_{\mathbb{C}}$. Because of the triple I, J and K we can consider E and E^* as two complex $GL(n, \mathbb{H})$ -representations endowed with their respective quaternionic structure maps defined by $x \mapsto \tilde{x} = Jx$ and $a_{\mathbb{C}} \mapsto \tilde{a}_{\mathbb{C}} = -(Ja)_{\mathbb{C}}$.

The actions of $GL(n, \mathbb{H}) Sp(1)$ on the real vector spaces \mathcal{V} and \mathcal{V}^* gives rise to $GL(n, \mathbb{H}) Sp(1)$ -isomorphisms which identify $\mathcal{V} \otimes_{\mathbb{R}} \mathbb{C} \cong E \otimes_{\mathbb{C}} H$ and $\mathcal{V}^* \otimes_{\mathbb{R}} \mathbb{C} \cong E^* \otimes_{\mathbb{C}} H^*$. In fact, such isomorphisms are defined respectively by $x \otimes_{\mathbb{R}} z \mapsto x \otimes_{\mathbb{C}} zh + Jx \otimes_{\mathbb{C}} z\tilde{h}$ and $a \otimes_{\mathbb{R}} z \mapsto a_{\mathbb{C}} \otimes_{\mathbb{C}} zh^{\omega} + (Ja)_{\mathbb{C}} \otimes_{\mathbb{C}} z\tilde{h}^{\omega}$, where we have fixed $h \in H$ such that $\langle h, h \rangle = 1$.

There are real structure maps defined on $E \otimes_{\mathbb{C}} H$ and $E^* \otimes_{\mathbb{C}} H^*$ which are given by $x \otimes_{\mathbb{C}} q \mapsto Jx \otimes_{\mathbb{C}} \tilde{q}$ and $a_{\mathbb{C}} \otimes_{\mathbb{C}} q^{\omega} \mapsto (Ja)_{\mathbb{C}} \otimes_{\mathbb{C}} \tilde{q}^{\omega}$, respectively. Such structure maps correspond to $x \otimes_{\mathbb{R}} z \mapsto x \otimes_{\mathbb{R}} \bar{z}$ and $a \otimes_{\mathbb{R}} z \mapsto a \otimes_{\mathbb{R}} \bar{z}$ respectively defined on $\mathcal{V} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathcal{V}^* \otimes_{\mathbb{R}} \mathbb{C}$. Thus, for the corresponding $(+1)$ -eigenspaces, we have $[EH] \cong \mathcal{V} \otimes_{\mathbb{R}} \mathbb{R} \cong \mathcal{V}$ and $[E^*H^*] \cong \mathcal{V}^* \otimes_{\mathbb{R}} \mathbb{R} \cong \mathcal{V}^*$.

For considering elements of the tensorial algebra of $[E^*H^*] \cong \mathcal{V}^*$, we need to compute the restrictions of $a_{\mathbb{C}}h^{\omega}$, $a_{\mathbb{C}}\tilde{h}^{\omega}$, $\tilde{a}_{\mathbb{C}}h^{\omega}$ and $\tilde{a}_{\mathbb{C}}\tilde{h}^{\omega}$ to $[EH] \cong \mathcal{V}$. Thus, for all $x \in \mathcal{V}$, we have $x \otimes h + Jx \otimes \tilde{h}$ which is the corresponding

element in $[EH]$ and obtain

$$\begin{aligned}
a_{\mathbb{C}} \otimes h^{\omega}(x \otimes h + Jx \otimes \tilde{h}) &= (Ja - iKa)(x), \\
\widetilde{a_{\mathbb{C}}} \otimes \tilde{h}^{\omega}(x \otimes h + Jx \otimes \tilde{h}) &= (-Ja - iKa)(x), \\
a_{\mathbb{C}} \otimes \tilde{h}^{\omega}(x \otimes h + Jx \otimes \tilde{h}) &= (a + iIa)(x), \\
\widetilde{a_{\mathbb{C}}} \otimes h^{\omega}(x \otimes h + Jx \otimes \tilde{h}) &= (a - iIa)(x).
\end{aligned} \tag{2.3}$$

3 Quaternionic decomposition of curvature

The space of Riemannian curvature tensors \mathcal{R} is the kernel of the map (2.1), which is $GL(n, \mathbb{H}) Sp(1)$ -equivariant, so \mathcal{R} is also a $GL(n, \mathbb{H}) Sp(1)$ -module. Our purpose here is to show the $GL(n, \mathbb{H}) Sp(1)$ -decomposition of \mathcal{R} into irreducible components.

On the one hand, the space $\Lambda^2 \mathcal{V}^*$ of skew-symmetric two-forms has the following decomposition into irreducible $GL(n, \mathbb{H}) Sp(1)$ -modules,

$$\Lambda^2 \mathcal{V}^* = S^2 E^* + \Lambda^2 E^* S^2 H, \tag{3.1}$$

where the real module $S^2 E^*$ is characterised as consisting of those $b \in \Lambda^2 \mathcal{V}^*$ such that $Ab = b$, for $A = I, J, K$, and the skew-symmetric two-forms $b \in \Lambda^2 E^* S^2 H$ are such that $\sum_{A=I,J,K} Ab = -b$.

Now, from (3.1) it follows

$$\begin{aligned}
S^2(\Lambda^2 \mathcal{V}^*) &= S^2(S^2 E^*) + S^2(\Lambda^2 E^* S^2 H) + S^2 E^* \Lambda^2 E^* S^2 H \\
&= S^2(S^2 E^*) + S^2(\Lambda^2 E^*)(S^4 H + \mathbb{R}) \\
&\quad + \Lambda^2(\Lambda^2 E^*) S^2 H + S^2 E^* \Lambda^2 E^* S^2 H,
\end{aligned}$$

where we have taken $S^2(S^2 H) \cong S^4 H + \mathbb{R}$ and $\Lambda^2(S^2 H) \cong S^2 H$ into account. Since $S^2(S^2 E^*) = S^4 E^* + U^{*22}$, $S^2(\Lambda^2 E^*) = \Lambda^4 E^* + U^{*22}$, $\Lambda^2(\Lambda^2 E^*) = U^{*211}$ and $S^2 E \Lambda^2 E^* = U^{*211} + U^{*31}$, we obtain

$$\begin{aligned}
S^2(\Lambda^2 \mathcal{V}^*) &= S^4 E^* + 2U^{*22} + \Lambda^4 E^* + (U^{*31} + 2U^{*211}) S^2 H \\
&\quad + (\Lambda^4 E^* + U^{*22}) S^4 H.
\end{aligned}$$

On the other hand, for the skew symmetric four-forms on \mathcal{V} , we obtain

$$\Lambda^4 \mathcal{V}^* = \Lambda^4 E^* S^4 H + U^{*211} S^2 H + U^{*22}.$$

Because there are non-vanishing values of the map (2.1) on each one of these three summands, we conclude that

$$\mathcal{R} = S^4 E^* + U^{*22} + \Lambda^4 E^* + (U^{*31} + U^{*211})S^2 H + U^{*22} S^4 H. \quad (3.2)$$

In order to give explicit descriptions for these modules, we will consider some $GL(n, \mathbb{H}) Sp(1)$ -endomorphisms on \mathcal{R} . The first one L is given by

$$L(R) = \sum_{\substack{1 \leq i < j \leq 4 \\ A=I, J, K}} A_{(i)} A_{(j)} R, \quad (3.3)$$

for all $R \in \mathcal{R}$. Regarding L , we have the following results.

Proposition 3.1. *The map L is $GL(n, \mathbb{H}) Sp(1)$ -equivariant and*

- (i) $S^4 E^* + U^{*22} + \Lambda^4 E^*$ consists of $R \in \mathcal{R}$ such that $L(R) = 6R$;
- (ii) $(U^{*31} + U^{*211})S^2 H$ consists of $R \in \mathcal{R}$ such that $L(R) = 2R$;
- (iii) $U^{*22} S^4 H$ consists of $R \in \mathcal{R}$ such that $L(R) = -6R$.

Proof. If we use another adapted basis I', J' and K' for \mathfrak{g} in equation (3.3), it is straightforward to check that we will obtain the same map L . Hence L is a $GL(n, \mathbb{H}) Sp(1)$ -map.

For (iii), we first show that we have the following decomposition of $S^2 H \otimes S^2 H$ into $Sp(1)$ -irreducible modules

$$S^2 H \otimes S^2 H = S^2(S^2 H) + \Lambda^2(S^2 H) = S^4 H + \mathbb{R} \omega_H \otimes \omega_H + S^2 H,$$

where we have taken $S^2(S^2 H) \cong S^4 H + \mathbb{R} \omega_H \otimes \omega_H$ and $\Lambda^2(S^2 H) \cong S^2 H$ into account.

Next, we consider $(a_{\mathbb{C}} b_{\mathbb{C}} c_{\mathbb{C}} d_{\mathbb{C}}) \tilde{h}^{\omega} \tilde{h}^{\omega} \tilde{h}^{\omega} \tilde{h}^{\omega} \in (\otimes^4 E^*) \otimes S^4 H \subset \otimes^4(E^* H)$, where we have omitted tensor product signs. Let $\Phi_1 \in [(\otimes^4 E) S^4 H] \subset \otimes^4[E^* H]$ be the tensor defined by $\Phi_1 = \text{Re}((a_{\mathbb{C}} \tilde{h}^{\omega})(b_{\mathbb{C}} \tilde{h}^{\omega})(c_{\mathbb{C}} \tilde{h}^{\omega})(d_{\mathbb{C}} \tilde{h}^{\omega})|_{\mathcal{V}})$, where Re means the real part. Now, using equations (2.3), we obtain

$$\begin{aligned} \Phi_1 = & abcd - aIbIcd - aIbcId - abIcId \\ & - IaIbcd - IabIcd - IabcId + IaIbIcId. \end{aligned}$$

From this last expression it is straightforward to check that $L(\Phi_1) = -6\Phi_1$. Since there are no conditions on a, b, c and d , part (iii) follows.

Part (i) follows by similar arguments considering $(a_{\mathbb{C}} b_{\mathbb{C}} c_{\mathbb{C}} d_{\mathbb{C}}) \omega_H \omega_H \in \otimes^4 E^* \subset \otimes^4(E^* H)$. Thus it is obtained $\Phi_2 \in [\otimes^4 E] \subset \otimes^4[E^* H]$ defined

by $\Phi_2 = \text{Re}((a_{\mathbb{C}}b_{\mathbb{C}}c_{\mathbb{C}}d_{\mathbb{C}})\omega_H\omega_{H|\mathcal{V}})$, we recall that $\omega_H = h^\omega\tilde{h}^\omega - \tilde{h}^\omega h^\omega$. After using equations (2.3), one can check that $L(\Phi_2) = 6\Phi_2$.

Finally, for part (ii), we recall that $(h^\omega h^\omega) \wedge (\tilde{h}^\omega \tilde{h}^\omega) \in \Lambda^2(S^2H) \cong S^2H$ and consider

$$(a_{\mathbb{C}}b_{\mathbb{C}}c_{\mathbb{C}}d_{\mathbb{C}})(h^\omega h^\omega \tilde{h}^\omega \tilde{h}^\omega - \tilde{h}^\omega \tilde{h}^\omega h^\omega h^\omega) \in (\otimes^4 E^*)S^2H \subset \otimes^4(E^*H),$$

then, for $\Phi_3 = \text{Re}((a_{\mathbb{C}}b_{\mathbb{C}}c_{\mathbb{C}}d_{\mathbb{C}})(h^\omega h^\omega \tilde{h}^\omega \tilde{h}^\omega - \tilde{h}^\omega \tilde{h}^\omega h^\omega h^\omega)_{|\mathcal{V}})$, one can check that $L(\Phi_3) = 2\Phi_3$. \square

In order to go further with the descriptions of the $GL(n, \mathbb{H}) Sp(1)$ -submodules of the space of curvature tensors \mathcal{R} , we will need to consider some $GL(n, \mathbb{H}) Sp(1)$ -maps from $\Lambda^2\mathcal{V}^* \otimes \Lambda^2\mathcal{V}^*$ to \mathcal{R} which are defined for $b, c \in \Lambda^2\mathcal{V}^*$ by:

$$\phi(b \otimes c) = 6b \odot c - b \wedge c, \quad (3.4)$$

$$\Phi(b \otimes c) = \sum_{A=I,J,K} (6(A_{(1)} + A_{(2)})b \odot (A_{(1)} + A_{(2)})c - (A_{(1)} + A_{(2)})b \wedge (A_{(1)} + A_{(2)})c), \quad (3.5)$$

$$\begin{aligned} \varphi(b \otimes c)(x, y, z, u) = \sum_{A=I,J,K} & ((A_{(1)} - A_{(2)})b(x, z)(A_{(1)} - A_{(2)})c(y, u) \\ & - (A_{(1)} - A_{(2)})b(x, u)(A_{(1)} - A_{(2)})c(y, z) \\ & + (A_{(1)} - A_{(2)})c(x, z)(A_{(1)} - A_{(2)})b(y, u) \\ & - (A_{(1)} - A_{(2)})c(x, u)(A_{(1)} - A_{(2)})b(y, z)), \end{aligned} \quad (3.6)$$

where we write $b \odot c = 1/2(b \otimes c + c \otimes b)$ and $x, y, z, u \in \mathcal{V}$. Note that the maps ϕ , φ and Φ vanish on $\Lambda^2(\Lambda^2\mathcal{V}^*)$, so we will consider them as maps $S^2(\Lambda^2\mathcal{V}^*) \rightarrow \mathcal{R}$.

Other $GL(n, \mathbb{H}) Sp(1)$ -maps that we will use are defined from $S^2\mathcal{V}^* \otimes S^2\mathcal{V}^*$ to \mathcal{R} . These maps are given for $b, c \in S^2\mathcal{V}^*$ by:

$$\begin{aligned} \psi(b \otimes c)(x, y, z, u) = & b(x, z)c(y, u) - b(x, u)c(y, z) \\ & + c(x, z)b(y, u) - c(x, u)b(y, z), \end{aligned} \quad (3.7)$$

$$\vartheta(b \otimes c) = \sum_{A=I,J,K} (6(A_{(1)} - A_{(2)})b \odot (A_{(1)} - A_{(2)})c - (A_{(1)} - A_{(2)})b \wedge (A_{(1)} - A_{(2)})c), \quad (3.8)$$

$$\begin{aligned} \Psi(b \otimes c)(x, y, z, u) = \sum_{A=I,J,K} & ((A_{(1)} + A_{(2)})b(x, z)(A_{(1)} + A_{(2)})c(y, u) \\ & - (A_{(1)} + A_{(2)})b(x, u)(A_{(1)} + A_{(2)})c(y, z) \\ & + (A_{(1)} + A_{(2)})c(x, z)(A_{(1)} + A_{(2)})b(y, u) \\ & - (A_{(1)} + A_{(2)})c(x, u)(A_{(1)} + A_{(2)})b(y, z)), \end{aligned} \quad (3.9)$$

for $x, y, z, u \in \mathcal{V}$. Analogously, since ψ , ϑ and Ψ vanish on $\Lambda^2(S^2\mathcal{V}^*)$, we will consider as defined $S^2(S^2\mathcal{V}^*) \rightarrow \mathcal{R}$.

Likewise, a fundamental tool that we will use to describe the irreducible $GL(n, \mathbb{H}) Sp(1)$ -modules of \mathcal{R} is the $GL(n, \mathbb{H}) Sp(1)$ -map $L_\sigma: \mathcal{R} \rightarrow \mathcal{R}$ which is defined by

$$\begin{aligned} L_\sigma(R) = \sum_{A=I,J,K} & (A_{(1)}A_{(2)} + A_{(2)}A_{(3)}\sigma + A_{(1)}A_{(3)}\sigma^2 \\ & + A_{(3)}A_{(4)} + A_{(1)}A_{(4)}\sigma + A_{(2)}A_{(4)}\sigma^2)R, \end{aligned} \quad (3.10)$$

where $\sigma = (123)$ is the permutation $1 \mapsto 2 \mapsto 3 \mapsto 1$ and $\sigma R(x, y, z, u) = R(z, x, y, u)$. As an illustration, $A_{(2)}A_{(3)}\sigma R(x, y, z, u) = R(Az, x, Ay, u)$.

Proposition 3.2. *For L and L_σ be as above, we have*

- (i) S^4E^* consists of $R \in \mathcal{R}$ such that $L(R) = 6R$ and $L_\sigma(R) = 12R$;
- (ii) U^{*22} consists of $R \in \mathcal{R}$ such that $L(R) = 6R$ and $L_\sigma(R) = 0$;
- (iii) Λ^4E^* consists of $R \in \mathcal{R}$ such that $L(R) = 6R$ and $L_\sigma(R) = -12R$;
- (iv) $U^{*31}S^2H$ consists of $R \in \mathcal{R}$ such that $L(R) = 2R$ and $L_\sigma(R) = 4R$;
- (v) $U^{*211}S^2H$ consists of $R \in \mathcal{R}$ such that $L(R) = 2R$ and $L_\sigma(R) = -4R$;
- (vi) For all $R \in U^{*22}S^4H$, $L(R) = -6R$ and $L_\sigma(R) = 0$.

Proof. For (i), (ii) and (iii), we consider $b_1, c_1 \in S^2E^* \subset \Lambda^2\mathcal{V}^*$. Using equations (3.4) and (3.6), it is straightforward to check

$$\begin{aligned} L_\sigma(4\phi(b_1 \odot c_1) + \varphi(b_1 \odot c_1)) &= 12(4\phi(b_1 \odot c_1) + \varphi(b_1 \odot c_1)), \\ L_\sigma(4\phi(b_1 \odot c_1) - \varphi(b_1 \odot c_1)) &= 0. \end{aligned}$$

Note that it is always possible to find b_1, c_1 such that $4\phi(b_1 \odot c_1) + \varphi(b_1 \odot c_1) \neq 0$ and $4\phi(b_1 \odot c_1) - \varphi(b_1 \odot c_1) \neq 0$.

Since $b_1 \odot c_1 \in S^2(S^2E^*) = S^4E^* + U^{*22}$, with Schur's Lemma in mind, then $4\phi(b_1 \odot c_1) + \varphi(b_1 \odot c_1) \in S^4E^*$ and $4\phi(b_1 \odot c_1) - \varphi(b_1 \odot c_1) \in U^{*22}$, or $4\phi(b_1 \odot c_1) + \varphi(b_1 \odot c_1) \in U^{*22}$ and $4\phi(b_1 \odot c_1) - \varphi(b_1 \odot c_1) \in S^4E^*$.

On the other hand, for $b_2, c_2 \in \Lambda^2 E \subset S^2 \mathcal{V}^*$, using equations (3.7) and (3.8), it is direct to check

$$\begin{aligned} L_\sigma(\vartheta(b_2 \odot c_2) + 12\psi(b_2 \odot c_2)) &= 0, \\ L_\sigma(\vartheta(b_2 \odot c_2) - 12\psi(b_2 \odot c_2)) &= -12(\vartheta(b_2 \odot c_2) - 12\psi(b_2 \odot c_2)). \end{aligned}$$

Likewise, $\vartheta(b_2 \odot c_2) + 12\psi(b_2 \odot c_2)$ and $\vartheta(b_2 \odot c_2) - 12\psi(b_2 \odot c_2)$ are not always vanished.

Since $b_2 \odot c_2 \in S^2(\Lambda^2 E^*) = \Lambda^4 E^* + U^{*22}$, we deduce that $\vartheta(b_2 \odot c_2) - 12\psi(b_2 \odot c_2) \in \Lambda^4 E^*$ and $\vartheta(b_2 \odot c_2) + 12\psi(b_2 \odot c_2) \in U^{*22}$, or $\vartheta(b_2 \odot c_2) - 12\psi(b_2 \odot c_2) \in U^{*22}$ and $\vartheta(b_2 \odot c_2) + 12\psi(b_2 \odot c_2) \in \Lambda^4 E^*$.

Therefore,

$$\begin{aligned} 4\phi(b_1 \odot c_1) + \varphi(b_1 \odot c_1) &\in S^4 E^*, \\ 4\phi(b_1 \odot c_1) - \varphi(b_1 \odot c_1), \vartheta(b_2 \odot c_2) + 12\psi(b_2 \odot c_2) &\in U^{*22}, \\ \vartheta(b_2 \odot c_2) - 12\psi(b_2 \odot c_2) &\in \Lambda^4 E^*. \end{aligned}$$

Thus, taking Proposition 3.1 into account, (i), (ii) and (iii) follow.

For (iv) and (v), we consider $b_3 \in S^2 E^* \subset \Lambda^2 \mathcal{V}^*$ and $c_3 \in \Lambda^2 E^* S^2 H$. Put $\alpha_1 = 4\phi(b_3 \odot c_3) + \varphi(b_3 \odot c_3)$ and $\alpha_2 = 4\phi(b_3 \odot c_3) - 3\varphi(b_3 \odot c_3)$. Then equations (3.4) and (3.6) give

$$L(\alpha_1) = 4\alpha_1 \quad \text{and} \quad L(\alpha_2) = -4\alpha_2,$$

so α_1 and α_2 belong to different irreducible summands of the space $S^2 E \otimes \Lambda^2 E^* S^2 H = U^{*211} S^2 H + U^{*31} S^2 H \subset S^2(\Lambda^2 \mathcal{V}^*)$ that contains $b_3 \odot c_3$. In Remark 4.4 below we will show that $\alpha_1 \in U^{*211} S^2 H$ and $\alpha_2 \in U^{*31} S^2 H$, proving (iv) and (v).

Finally, (vi) will be proved below, see Remark 4.7. \square

4 Almost quaternion-Hermitian decomposition of curvature

In §3, using the action of the Lie group $GL(n, \mathbb{H}) Sp(1)$ we obtained and described the decomposition of the space of curvature tensors \mathcal{R} given by equation (3.2). In this section we will study the decompositions each one of these submodules under the action of the subgroup $Sp(n) Sp(1)$ of $GL(n, \mathbb{H}) Sp(1)$.

As we have pointed out above, the main difference between $Sp(n)$ and $GL(n, \mathbb{H})$ is that $Sp(n)$ preserves the complex symplectic form ω_E . Moreover, we have an identification $E \cong E^*$ by ω_E and, consequently, all tensor modules are identified with their corresponding duals. Therefore, we will write

$$\mathcal{R} = S^4E + U^{22} + \Lambda^4E + (U^{31} + U^{211})S^2H + U^{22}S^4H.$$

On the other hand, the presence of the metric $g = \langle \cdot, \cdot \rangle$ allows to work with the Ricci and q-Ricci curvature tensors. Now we show the relationships of these tensors with the maps L and L_σ .

Lemma 4.1. *If L and L_σ are the $Sp(n) Sp(1)$ -maps defined by equations (3.3) and (3.10), respectively, then*

$$\begin{aligned} \text{Ric}(L(R))(X, Y) &= 3 \text{Ric}(X, Y) + \sum_{A=I, J, K} \text{Ric}(AX, AY), \\ \text{Ric}^q(L(R))(X, Y) &= 3 \text{Ric}^q(X, Y) + \sum_{A=I, J, K} \text{Ric}^q(AX, AY), \\ \text{Ric}(L_\sigma(R))(X, Y) &= 3 \text{Ric}^q(X, Y) + 3 \text{Ric}^q(Y, X) - 3 \text{Ric}(X, Y) \\ &\quad - \sum_{A=I, J, K} \text{Ric}(AX, AY). \end{aligned}$$

Proof. It follows by straightforward computation. \square

Now we will analyse the behaviour of the different $GL(n, \mathbb{H}) Sp(1)$ -submodules of \mathcal{R} under the action of $Sp(n) Sp(1)$. Since contractions by ω_E on the $Sp(n)$ -module S^4E are all zero, S^4E is also irreducible as an $Sp(n)$ -module. Therefore, $S^4E \cong S^4E \otimes \mathbb{C}(\omega_H \otimes \omega_H)$ is irreducible as an $Sp(n) Sp(1)$ -module.

To study U^{22} , we consider $U^{22} \subset S^2(\Lambda^2E)$ and the map $\omega_{34}: U^{22} \rightarrow E \otimes E$ given by contraction with ω_E on the $(3, 4)$ -indices. One has that $\omega_{34}(U^{22}) = \Lambda^2E$ and we write $V^{22} = \ker \omega_{34}$. Therefore, $U^{22} = V^{22} + \Lambda_0^2E + \mathbb{C}\omega_E \otimes \omega_E$ is the decomposition U^{22} into $Sp(n)$ -irreducible modules. Then, for U^{22} as submodule of \mathcal{R} and $n > 1$, we have the following decomposition into $Sp(n) Sp(1)$ -irreducible summands,

$$U^{22} = V^{22} + (\Lambda_0^2E)_a + \mathbb{R}_a,$$

where we have inserted the subscript a to distinguish these modules from other copies of Λ_0^2E and \mathbb{R} in \mathcal{R} . When $n > 1$, the three summands of the

decomposition of U^{22} are non-zero. However, if $n = 1$, from $S^2(S^2E) = S^4E + U^{22}$, we have $\dim U^{22} = 1$. Therefore, for $n = 1$,

$$U^{22} = \mathbb{R}_a.$$

For providing detailed descriptions of these modules, in the next Proposition we will need to consider

$$\begin{aligned}\pi_1(x, y, z, u) &= \langle x, z \rangle \langle y, u \rangle - \langle x, u \rangle \langle y, z \rangle, \\ \pi_2 &= \sum_{A=I, J, K} (6\omega_A \odot \omega_A - \omega_A \wedge \omega_A).\end{aligned}$$

Proposition 4.2. *Let ϑ and ψ be the maps respectively defined by equations (3.7) and (3.8), then*

- (i) V^{22} consists of $R \in \mathcal{R}$ such that $L(R) = 6R$, $L_\sigma(R) = 0$ and $\text{Ric}(R) = 0$;
- (ii) $(\Lambda_0^2 E)_a$ consists of $R = \vartheta(b \otimes g) + 12\psi(b \otimes g)$, where $b \in \Lambda_0^2 E \subset S^2\mathcal{V}^*$ and $g = \langle \cdot, \cdot \rangle$ is the metric. Moreover, $\text{Ric}(R) = 48(n+1)b$;
- (iii) $\mathbb{R}_a = \mathbb{R}(\vartheta(g \otimes g) + 12\psi(g \otimes g)) = \mathbb{R}(\pi_2 + 6\pi_1)$. Moreover, $\text{Ric}(\pi_2 + 6\pi_1) = 12(2n+1)g$.
- (iv) If $R \in V^{22} + (\Lambda_0^2 E)_a + \mathbb{R}_a$, then $\text{Ric} = \text{Ric}^a \in \Lambda_0^2 E + \mathbb{R}g$, i.e., for $A = I, J, K$, $A \text{ Ric} = \text{Ric}$.

Proof. This follows from Propositions 3.1 and 3.2, the considerations in the proof of Proposition 3.2, and the facts $\psi(g \otimes g) = 2\pi_1$ and $\vartheta(g \otimes g) = 4\pi_2$. Part (iv) is a consequence of Proposition 3.2(ii) and Lemma 4.1. \square

Now let us consider the module $\Lambda^4 E$. By contracting with ω_E , we obtain the decomposition into irreducible $Sp(n)$ -modules given by $\Lambda^4 E = \Lambda_0^4 E + \omega_E \wedge \Lambda_0^2 E + \mathbb{C}(\omega_E \wedge \omega_E)$. Thus it follows that the decomposition of $\Lambda^4 E \subset \mathcal{R}$ into irreducible $Sp(n) Sp(1)$ -modules is given by

$$\Lambda^4 E = \Lambda_0^4 E + (\Lambda_0^2 E)_b + \mathbb{R}_b. \quad (4.1)$$

Note that:

- if $n > 3$, then each one of the three summands is non-zero;
- if $n = 3$, then $\Lambda^4 E = (\Lambda_0^2 E)_b + \mathbb{R}_b$;
- if $n = 2$, then $\Lambda^4 E = \mathbb{R}_b$; and
- if $n = 1$, then $\Lambda^4 E = \{0\}$.

Next we give more details relative to summands in the right side of equation (4.1).

Proposition 4.3. *Let ϑ and ψ be the maps defined respectively by equations (3.7) and (3.8), then*

- (i) $\Lambda_0^4 E$ consists of $R \in \mathcal{R}$ such that $L(R) = 6R$, $L_\sigma(R) = -12R$ and $\text{Ric}(R) = 0$;
- (ii) $(\Lambda_0^2 E)_b$ consists of $R = \vartheta(b \otimes g) - 12\psi(b \otimes g)$, where $b \in \Lambda_0^2 E \subset S^2 \mathcal{V}^*$. Moreover, $\text{Ric}(R) = -48(n-2)b$;
- (iii) $\mathbb{R}_b = \mathbb{R}(\vartheta(g \otimes g) - 12\psi(g \otimes g)) = \mathbb{R}(\pi_2 - 6\pi_1)$. Moreover, $\text{Ric}(\pi_2 - 6\pi_1) = -24(n-1)g$;
- (iv) If $R \in \Lambda_0^4 E + (\Lambda_0^2 E)_b + \mathbb{R}_b$, then $\text{Ric} = -\text{Ric}^q \in \Lambda_0^2 E + \mathbb{R}g$, i.e., for $A = I, J, K$, $A \text{ Ric} = \text{Ric}$.

Proof. This follows from Propositions 3.1 and 3.2, considerations contained in the proof of Proposition 3.2 and Lemma 4.1. \square

We have already pointed out that $S^2 E \Lambda^2 E = U^{31} + U^{211}$. Moreover, one can check that $U^{31} = (S^3 E \otimes E) \cap (S^2 E \otimes \Lambda^2 E)$ and $U^{211} = (E \otimes \Lambda^3 E) \cap (S^2 E \otimes \Lambda^2 E)$. Therefore, contracting with ω_E , one obtains the following decompositions into irreducible $Sp(n)$ -summands $U^{31} = V^{31} + S^2 E$ and $U^{211} = V^{211} + S^2 E + \Lambda_0^2 E$. Thus, for the modules $U^{31} S^2 H, U^{211} S^2 H \subset \mathcal{R}$, we have the following decompositions into irreducible $Sp(n) Sp(1)$ -modules,

$$U^{31} S^2 H = V^{31} S^2 H + (S^2 E S^2 H)_a, \quad (4.2)$$

$$U^{211} S^2 H = V^{211} S^2 H + (S^2 E S^2 H)_b + \Lambda_0^2 E S^2 H. \quad (4.3)$$

All of this happens for high dimensions. However for low dimensions some particular cases must be pointed out.

– For U^{31} . If $n > 1$, the two summands V^{31} and $S^2 E$ are non-zero. If $n = 1$, then $\Lambda^2 E = \mathbb{R}\omega_E$ and $U^{31} = S^2 E$. Therefore, for $n = 1$, we have

$$U^{31} S^2 H = (S^2 E S^2 H)_a.$$

– For U^{211} . If $n > 2$, the three summands V^{211} , $S^2 E$ and $\Lambda_0^2 E$ are non-zero. If $n = 2$, then $\Lambda^3 E = E \wedge \omega_E$ and $U^{211} = S^2 E + \Lambda_0^2 E$. Therefore, for $n = 2$, we have

$$U^{211} S^2 H = (S^2 E S^2 H)_b + \Lambda_0^2 E S^2 H.$$

If $n = 1$, then $\Lambda^3 E = \{0\}$. Therefore, $U^{211} = \{0\}$ and $U^{211} S^2 H = \{0\}$.

Remark 4.4. At this point we can complete the proof of parts (iv) and (v) of Proposition 3.2. In fact, for $b \in S^2E \subset \Lambda^2\mathcal{V}^*$ and $c \in \Lambda^2ES^2H \subset \Lambda^2\mathcal{V}^*$, it is straightforward to check

$$\text{Ric}^q(4\phi(b \odot c) + \varphi(b \odot c)) = 16e_i \lrcorner b \odot e_i \lrcorner c - 8 \sum_{A=I,J,K} \langle \omega_A, c \rangle A_{(1)} b,$$

where \lrcorner denotes contraction. Thus, we have $\text{Ric}^q(4\phi(b \odot c) + \varphi(b \odot c)) \in S^2ES^2H$.

On the other hand,

$$\begin{aligned} \text{Ric}^q(4\phi(b \odot c) - 3\varphi(b \odot c)) &= -40e_i \lrcorner b \otimes e_i \lrcorner c + 24e_i \lrcorner c \otimes e_i \lrcorner b \\ &\quad - 8 \sum_{A=I,J,K} \langle \omega_A, c \rangle A_{(1)} b, \end{aligned}$$

which can have non-zero components in both S^2ES^2H and $\Lambda_0^2ES^2H$. Thus $\text{Ric}^q(4\phi(b \odot c) - 3\varphi(b \odot c)) \in S^2ES^2H + \Lambda_0^2ES^2H$. All of this, taking equations (4.2) and (4.3) into account, implies $4\phi(b \odot c) + 3\varphi(b \odot c) \in U^{31}S^2H$ and $4\phi(b \odot c) - 3\varphi(b \odot c) \in U^{211}S^2H$.

Now, we show more details for the summands of equations (4.2) and (4.3).

Proposition 4.5. *Let ϑ and ψ be the maps defined respectively by equations (3.7) and (3.8), then*

- (i) $V^{31}S^2H$ consists of $R \in \mathcal{R}$ such that $L(R) = 2R$, $L_\sigma(R) = 4R$ and $\text{Ric}(R) = 0$;
- (ii) $(S^2ES^2H)_a$ consists of $R = \vartheta(b \otimes g) + 4\psi(b \otimes g)$, where $b \in S^2ES^2H \subset S^2\mathcal{V}^*$. Moreover, $\text{Ric}(R) = 16(n+1)b$;
- (iii) $V^{211}S^2H$ consists of $R \in \mathcal{R}$ such that $L(R) = 2R$, $L_\sigma(R) = -4R$ and $\text{Ric}(R) = 0$;
- (iv) $(S^2ES^2H)_b$ consists of $R = \vartheta(b \otimes g) - 12\psi(b \otimes g)$, where $b \in S^2ES^2H \subset S^2\mathcal{V}^*$. Moreover, $\text{Ric}(R) = -48(n-1)b$;
- (v) $\Lambda_0^2ES^2H$ consists of R such that

$$R = \sum_{A=I,J,K} (6(A_{(1)} + A_{(2)})b \odot \omega_A - (A_{(1)} + A_{(2)})b \wedge \omega_A),$$

where $b \in \Lambda_0^2ES^2H \subset \Lambda^2\mathcal{V}^*$. Moreover, $\text{Ric}^q(R) = -16nb$;

- (vi) if $R \in V^{31}S^2H + (S^2ES^2H)_a$, then $\text{Ric} = \text{Ric}^q \in S^2ES^2H$;
- (vii) if $R \in V^{211}S^2H + (S^2ES^2H)_b + \Lambda_0^2ES^2H$ and Ric_s^q denotes the symmetric part of Ric^q , then $\text{Ric} = -3\text{Ric}_s^q \in S^2ES^2H$.

Proof. For $b \in S^2ES^2H \subset S^2\mathcal{V}^*$, it is not hard to check that

$$\begin{aligned} L_\sigma(\vartheta(b \otimes g) + 4\psi(b \otimes g)) &= 4(\vartheta(b \otimes g) + 4\psi(b \otimes g)), \\ L_\sigma(\vartheta(b \otimes g) - 12\psi(b \otimes g)) &= -4(\vartheta(b \otimes g) - 12\psi(b \otimes g)). \end{aligned}$$

Now, all parts follow from Propositions 3.1 and 3.2 and Lemma 4.1. \square

Since we have already shown the $Sp(n)$ -decomposition $U^{22} = V^{22} + \Lambda_0^2 E + \mathbb{C}\omega_E \otimes \omega_E$, then, for $U^{22}S^4H \subset \mathcal{R}$ and $n > 1$, we obtain

$$U^{22}S^4H = V^{22}S^4H + \Lambda_0^2 ES^4H + S^4H.$$

For $n = 1$, as it was above pointed out, $U^{22} = \mathbb{C}$. Therefore, for $n = 1$, we have

$$U^{22}S^4H = S^4H.$$

Proposition 4.6. (i) $V^{22}S^4H$ consists of $R \in \mathcal{R}$ such that $L(R) = -6R$ and, for $A = I, J, K$, $\text{Ric}_A^*(R) = 0$;

(ii) $\Lambda_0^2 ES^4H$ consists of R such that

$$R = \sum_{A=I,J,K} (6b_A \odot \omega_A - b_A \wedge \omega_A), \quad (4.4)$$

where $b_I, b_J, b_K \in \Lambda_0^2 ES^2H \subset \Lambda^2\mathcal{V}^*$ are such that $\sum_{A=I,J,K} A_{(2)}b_A = 0$;

(iii) S^4H consists of R such that

$$R = \sum_{A=I,J,K} (6b_A \odot \omega_A - b_A \wedge \omega_A), \quad (4.5)$$

where $b_I, b_J, b_K \in S^2H \subset \Lambda^2\mathcal{V}^*$ are such that $\sum_{A=I,J,K} A_{(2)}b_A = 0$.

Proof. For (ii), if R is given by equation (4.4), it is straightforward to check $L(R) = -6R$. On the other hand, it is not hard to obtain, for $A = I, J, K$,

$$\text{Ric}_A^* = -4(n+1)A_{(2)}b_A \in S^2ES^2H + \Lambda_0^2 ES^2H.$$

Hence $\text{Ric}^q = 0$, but the local Ricci tensors Ric_A^* are not necessarily zero.

For (iii), if R is given by equation (4.5), where $b_I = \lambda_{II}\omega_I + \lambda_{JI}\omega_J + \lambda_{KI}\omega_K$. It is also straightforward to check $L(R) = -6R$. In this case, we have

$$\text{Ric}_I^* = 4(n+1)(\lambda_{II}g + \lambda_{KI}\omega_J - \lambda_{JI}\omega_K) \in \mathbb{R}g + S^2H,$$

and also $\text{Ric}^q = 0$. Since there are curvature tensors in the conditions of (i), part (i) follows. \square

Remark 4.7. Now we will prove that $L_\sigma(R) = 0$, for all $R \in U^{*22}S^4H$. In fact, we consider $R_1 = 6\omega_I \odot \omega_J - \omega_I \wedge \omega_J \in S^4H \subset U^{*22}S^4H$. It is direct to check $L_\sigma(R_1) = 0$. By Schur's Lemma, the assertion follows.

Remark 4.8. For a fixed adapted basis I, J, K of \mathcal{G} , if $R \in \Lambda_0^2 ES^4 + S^4H$, then R is determined by a unique triple b_I, b_J, b_K . In (iii) of Proposition 4.6, we can write the condition $\sum_{A=I,J,K} \langle b_A, \omega_A \rangle = 0$ instead of $\sum_{A=I,J,K} A_{(2)} b_A = 0$, but in such a case more than one triple b_I, b_J, b_K can determine the same element of S^4H .

In summary, relative to the $Sp(n)Sp(1)$ -decomposition of the space of Riemannian curvature tensors \mathcal{R} , we have the following cases:

- if the dimension of \mathcal{V} is strictly greater than 12, $n > 3$, then

$$\begin{aligned} \mathcal{R} = & S^4E + (\mathbb{R}_a + (\Lambda_0^2 E)_a + V^{22}) + (\mathbb{R}_b + (\Lambda_0^2 E)_b + \Lambda_0^4 E) \\ & + ((S^2 ES^2 H)_a + V^{31} S^2 H) + ((S^2 ES^2 H)_b + \Lambda_0^2 ES^2 H + V^{211} S^2 H) \\ & + (S^4 H + \Lambda_0^2 ES^4 H + V^{22} S^4 H); \end{aligned}$$

- if the dimension of \mathcal{V} is 12, $n = 3$, then

$$\begin{aligned} \mathcal{R} = & S^4E + (\mathbb{R}_a + (\Lambda_0^2 E)_a + V^{22}) + (\mathbb{R}_b + (\Lambda_0^2 E)_b) \\ & + ((S^2 ES^2 H)_a + V^{31} S^2 H) + ((S^2 ES^2 H)_b + \Lambda_0^2 ES^2 H + V^{211} S^2 H) \\ & + (S^4 H + \Lambda_0^2 ES^4 H + V^{22} S^4 H); \end{aligned}$$

- if the dimension of \mathcal{V} is 8, $n = 2$, then

$$\begin{aligned} \mathcal{R} = & S^4E + (\mathbb{R}_a + (\Lambda_0^2 E)_a + V^{22}) + \mathbb{R}_b \\ & + ((S^2 ES^2 H)_a + V^{31} S^2 H) + ((S^2 ES^2 H)_b + \Lambda_0^2 ES^2 H) \\ & + (S^4 H + \Lambda_0^2 ES^4 H + V^{22} S^4 H); \end{aligned}$$

- and, if the dimension of \mathcal{V} is 4, $n = 1$, then

$$\mathcal{R} = S^4E + \mathbb{R}_a + (S^2 ES^2 H)_a + S^4H.$$

5 Intrinsic torsion

Let G be a subgroup of the linear group $GL(m, \mathbb{R})$. A manifold M is said to be equipped with a G -structure, if there is a principal G -subbundle $P \rightarrow M$ of the principal frame bundle. In this situation, there always exist connections,

called *G-connections*, defined on the subbundle P . Moreover, if $(M^m, g = \langle \cdot, \cdot \rangle)$ is an orientable m -dimensional Riemannian manifold and G is a closed and connected subgroup of $SO(m)$, then there exists a unique metric G -connection $\tilde{\nabla}$ such that $\xi_x = \tilde{\nabla}_x - \nabla_x$ takes its values in \mathfrak{g}^\perp , where \mathfrak{g}^\perp denotes the orthogonal complement in $\mathfrak{so}(m)$ of the Lie algebra \mathfrak{g} of G and ∇ denotes the Levi-Civita connection [15, 6]. The tensor ξ is the *intrinsic torsion* of the G -structure and $\tilde{\nabla}$ is called the *minimal G-connection*.

A $4n$ -dimensional manifold M , $n > 1$, is said to be *almost quaternion-Hermitian*, if M is equipped with an $Sp(n)Sp(1)$ -structure. This is equivalent to the presence of a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and a rank-three subbundle \mathcal{G} of the endomorphism bundle $\text{End } TM$, such that locally \mathcal{G} has an *adapted basis* I, J, K satisfying $I^2 = J^2 = -1$ and $K = IJ = -JI$, and $\langle AX, AY \rangle = \langle X, Y \rangle$, for all $X, Y \in T_x M$ and $A = I, J, K$. An almost quaternion-Hermitian manifold with a global adapted basis is called an *almost hyperHermitian* manifold. The manifold is then equipped with an $Sp(n)$ -structure.

On each point p of these manifolds, the tangent space $T_p M$ can be identified with the vector space \mathcal{V} of the previous section. Thus there are three local Kähler-forms $\omega_A(X, Y) = \langle X, AY \rangle$, $A = I, J, K$. From these one may define a global, non-degenerate four-form Ω , the *fundamental form*, by the local formula (2.2).

In this section, we will recall some information about the intrinsic torsion of almost quaternion-Hermitian manifolds. More details may be found in [12], where it is also explained how to explicitly compute the intrinsic torsion via the exterior algebra.

A connection $\tilde{\nabla}$ is an $Sp(n)Sp(1)$ -connection, if $\tilde{\nabla}\Omega = 0$ or, equivalently, if for any point of the manifold there exists a local adapted basis I, J, K of \mathcal{G} such that

$$\begin{aligned} (\tilde{\nabla}_X I)Y &= \lambda_K(X)JY - \lambda_J(X)KY, \\ (\tilde{\nabla}_X J)Y &= \lambda_I(X)KY - \lambda_K(X)IY, \\ (\tilde{\nabla}_X K)Y &= \lambda_J(X)IY - \lambda_I(X)JY. \end{aligned}$$

With respect to the Levi-Civita connection one then has formulæ such as

$$(\nabla_X I)Y = \lambda_K(X)JY - \lambda_J(X)KY - \xi_X IY + I\xi_X Y. \quad (5.1)$$

Proposition 5.1 (Cabrera & Swann [12]). *The minimal $Sp(n)Sp(1)$ -connection is given by $\tilde{\nabla} = \nabla + \xi$, where ∇ is the Levi-Civita connection and*

the intrinsic $Sp(n) Sp(1)$ -torsion ξ is defined by

$$\xi_X Y = -\frac{1}{4} \sum_{A=I,J,K} A(\nabla_X A)Y + \frac{1}{2} \sum_{A=I,J,K} \lambda_A(X)AY,$$

for all vectors X, Y , being the one-forms λ_I , λ_J and λ_K defined by cyclically permuting I, J, K in the expression

$$\lambda_I(X) = \frac{1}{2n} \langle \nabla_X \omega_J, \omega_K \rangle.$$

The next result describes the decomposition of the space $T^*M \otimes \Lambda_0^2 ES^2H$ of possible intrinsic torsion tensors into irreducible $Sp(n) Sp(1)$ -modules.

Proposition 5.2 (Swann [16]). *The intrinsic torsion ξ of an almost quaternion-Hermitian manifold M of dimension at least 8, has the property*

$$\xi \in T^*M \otimes \Lambda_0^2 ES^2H = (\Lambda_0^3 E + K + E)(S^3H + H).$$

If the dimension of M is at least 12, all the modules of the sum are non-zero. For an eight-dimensional manifold M , we have $\Lambda_0^3 ES^3H = \Lambda_0^3 EH = \{0\}$. Therefore, for $\dim M \geq 12$ and $\dim M = 8$, we have respectively $2^6 = 64$ and $2^4 = 16$ classes of almost quaternion-Hermitian manifolds. Explicit conditions characterising these classes can be found in [13].

We use this Proposition to decompose ξ as

$$\xi = \xi_{33} + \xi_{K3} + \xi_{E3} + \xi_{3H} + \xi_{KH} + \xi_{EH},$$

where $\xi_{UF} \in U \otimes F$, for $U = \Lambda_0^3 E, K, E$ and $F = S^3H, H$. The components of the intrinsic torsion ξ have the following specific symmetry properties and characterisations described in [12].

(i) ξ_{33} is a tensor characterised by the conditions:

- (a) $\sum_{A=I,J,K} (\xi_{33})_A A = -\sum_{A=I,J,K} A(\xi_{33})_A = -\xi_{33}$,
- (b) $\langle \cdot, (\xi_{33}) \cdot \cdot \rangle$ is a skew-symmetric three-form.

(ii) ξ_{K3} is a tensor characterised by the conditions:

- (a) $\sum_{A=I,J,K} (\xi_{K3})_A A = -\sum_{A=I,J,K} A(\xi_{K3})_A = -\xi_{K3}$,
- (b) $\mathfrak{S}_{XYZ} \langle Y, (\xi_{K3})_X Z \rangle = 0$.

(iii) ξ_{E3} is given by

$$\begin{aligned} \langle Y, (\xi_{E3})_X Z \rangle \\ = \frac{1}{n} \sum_{A=I,J,K} (nA(\theta_A^\xi - \theta^\xi) \wedge \omega_A - (n-1)A(\theta_A^\xi - \theta^\xi) \otimes \omega_A)(X, Y, Z), \end{aligned}$$

where θ^ξ is the one-form defined by

$$\frac{6}{n}(2n+1)(n-1)\theta^\xi(X) = -\langle \xi_{e_i} e_i, X \rangle = - \sum_{A=I,J,K} \langle A\xi_{e_i} A e_i, X \rangle, \quad (5.2)$$

and $\theta_I^\xi, \theta_J^\xi, \theta_K^\xi$ are the local one-forms given by

$$\frac{2}{n}(2n+1)(n-1)\theta_A^\xi(X) = -\langle A\xi_{e_i} A e_i, X \rangle.$$

Note that $3\theta^\xi = \theta_I^\xi + \theta_J^\xi + \theta_K^\xi$.

(iv) ξ_{3H} is a tensor characterised by the conditions:

- (a) $(\xi_{3H})_A A - A(\xi_{3H})_A - A\xi_{3H} A = \xi_{3H}$, for $A = I, J, K$,
- (b) $\mathfrak{S}_{X,Y,Z} \langle Y, (\xi_{3H})_X Z \rangle = 0$.

(v) ξ_{KH} is a tensor characterised by the conditions:

- (a) $(\xi_{KH})_A A - A(\xi_{KH})_A - A\xi_{KH} A = \xi_{KH}$, for $A = I, J, K$;
- (b) there exists a skew-symmetric three-form $\psi^{(K)}$ such that

$$\langle Y, (\xi_{KH})_X Z \rangle = (3\psi^{(K)} - \sum_{A=I,J,K} A_{(23)} \psi^{(K)})(X, Y, Z);$$

- (c) $\sum_{i=1}^{4n} (\xi_{KH})_{e_i} e_i = 0$.

(vi) ξ_{EH} is given by

$$\begin{aligned} \langle Y, (\xi_{EH})_X Z \rangle &= 3e_i \otimes e_i \wedge \theta^\xi(X, Y, Z) \\ &\quad - \sum_{A=I,J,K} (e_i \otimes A e_i \wedge A \theta^\xi + \frac{2}{n} A \theta^\xi \otimes \omega_A)(X, Y, Z), \end{aligned}$$

where θ^ξ is the global one-form defined by (5.2).

(vii) The part $\xi_{S^3H} = \xi_{33} + \xi_{K3} + \xi_{E3}$ of ξ in $(\Lambda_0^2 E + K + E)S^3H$ is characterised by the condition

$$\sum_{A=I,J,K} (\xi_{S^3H})_A A = - \sum_{A=I,J,K} A(\xi_{S^3H})_A = -\xi_{S^3H}.$$

(viii) The part $\xi_H = \xi_{3H} + \xi_{KH} + \xi_{EH}$ of ξ in $(\Lambda_0^2 E + K + E)H$ is characterised by the condition

$$(\xi_H)_A A - A(\xi_H)_A - A(\xi_H)A = \xi_H,$$

for $A = I, J, K$.

6 Curvature and intrinsic torsion

In order to study the contribution of the intrinsic $Sp(n)$ $Sp(1)$ -torsion to the different components of the Riemannian curvature tensor, we consider the $Sp(n)$ $Sp(1)$ -map $\pi_{1es}: \Lambda^2 T^* M \otimes \Lambda^2 T^* M \rightarrow \Lambda^2 T^* M \otimes \Lambda^2 ES^2 H$ defined by

$$4\pi_{1es}(a) = 3a - \sum_{A=I,J,K} A_{(3)} A_{(4)} a.$$

Let $\tilde{\mathbf{a}}: T^* M \otimes T^* M \otimes \text{End } T^* M \rightarrow \Lambda^2 T^* M \otimes \text{End } T^* M$ be the skewing map and define $\tilde{\mathbf{b}}: (T^* M \otimes \text{End } T^* M) \otimes (T^* M \otimes \text{End } T^* M) \rightarrow \Lambda^2 T^* M \otimes \text{End } T^* M$ by $\tilde{\mathbf{b}}(\xi \otimes \zeta)_{X,Y} Z = \xi_{\zeta_X Y} Z - \xi_{\zeta_Y X} Z$.

Lemma 6.1. *For the curvature tensor $R \in \mathcal{R}$, the intrinsic $Sp(n)$ $Sp(1)$ -torsion ξ and $\gamma_I = d\lambda_I + \lambda_J \wedge \lambda_K$, we have*

$$\begin{aligned} \pi_{1es}(R)(X, Y, Z, U) &= \frac{1}{2} \sum_{A=I,J,K} \gamma_A \otimes \omega_A(X, Y, Z, U) + \langle \tilde{\mathbf{a}}(\tilde{\nabla} \xi)_{X,Y} Z, U \rangle \\ &\quad - \frac{3}{4} \langle \tilde{\mathbf{a}}(\xi \circ \xi)_{X,Y} Z, U \rangle - \frac{1}{4} \sum_{A=I,J,K} \langle A \tilde{\mathbf{a}}(\xi \circ \xi)_{X,Y} A Z, U \rangle \\ &\quad + \langle \tilde{\mathbf{b}}(\xi \otimes \xi)_{X,Y} Z, U \rangle. \end{aligned}$$

Proof. Since $R(X, Y, IZ, IU) - R(X, Y, Z, U) = -(R_{X,Y} \omega_I)(Z, IU)$, using the so-called Ricci formula [3, p. 26], we have

$$R(X, Y, IZ, IU) - R(X, Y, Z, U) = \tilde{\mathbf{a}}(\nabla^2 \omega_I)_{X,Y}(Z, IU), \quad (6.1)$$

where in this case $\tilde{\mathbf{a}}: T^* M \otimes T^* M \otimes \Lambda^2 T^* M \rightarrow \Lambda^2 T^* M \otimes \Lambda^2 T^* M$ is also the skewing map. On the other hand, from equation (5.1), it follows

$$\begin{aligned} (\nabla_X \omega_I)(Y, Z) &= \lambda_K(X) \omega_J(Y, Z) - \lambda_J(X) \omega_K(Y, Z) \\ &\quad - \langle Y, \xi_X IZ \rangle + \langle Y, I \xi_X Z \rangle. \end{aligned} \quad (6.2)$$

Now, taking $\tilde{\nabla} = \nabla + \xi$ into account and using repeatedly equation (6.2), from the right side of equation (6.1) we get

$$\begin{aligned}
(1 - I_{(3)}I_{(4)})R(X, Y, Z, U) &= (\gamma_J \otimes \omega_J + \gamma_K \otimes \omega_K)(X, Y, Z, U) \\
&\quad + 2\langle \tilde{\mathbf{a}}(\lambda_J \otimes K\xi I)_{X,Y}Z, U \rangle - 2\langle \tilde{\mathbf{a}}(\lambda_K \otimes I\xi J)_{X,Y}Z, U \rangle \\
&\quad + \langle \tilde{\mathbf{a}}(\lambda_J \otimes \xi J)_{X,Y}Z, U \rangle + \langle \tilde{\mathbf{a}}(\lambda_J \otimes J\xi)_{X,Y}Z, U \rangle \\
&\quad - \langle \tilde{\mathbf{a}}(\lambda_K \otimes \xi K)_{X,Y}Z, U \rangle - \langle \tilde{\mathbf{a}}(\lambda_K \otimes K\xi)_{X,Y}Z, U \rangle \\
&\quad + \langle \tilde{\mathbf{a}}(\tilde{\nabla}\xi)_{X,Y}Z, U \rangle + \langle \tilde{\mathbf{a}}(\tilde{\nabla}I\xi I)_{X,Y}Z, U \rangle \\
&\quad - \langle \tilde{\mathbf{a}}(\xi \circ \xi)_{X,Y}Z, U \rangle - \langle I\tilde{\mathbf{a}}(\xi \circ \xi)_{X,Y}IZ, U \rangle \\
&\quad + \langle \tilde{\mathbf{b}}(\xi \otimes \xi)_{X,Y}Z, U \rangle + \langle I\tilde{\mathbf{b}}(\xi \otimes \xi)_{X,Y}IZ, U \rangle.
\end{aligned} \tag{6.3}$$

From this identity the Lemma follows. \square

Another projection that we need to consider is $\pi_{1s}: \Lambda^2 T^*M \otimes \Lambda^2 T^*M \rightarrow \Lambda^2 T^*M \otimes S^2 H$ given by

$$\pi_{1s}(a)(X, Y, Z, U) = \frac{1}{4n} \sum_{A=I,J,K} \langle a(X, Y, \cdot, \cdot), \omega_A \rangle \omega_A(Z, U).$$

Lemma 6.2. *For the curvature tensor $R \in \mathcal{R}$, the intrinsic $Sp(n)Sp(1)$ -torsion ξ and $\gamma_I = d\lambda_I + \lambda_J \wedge \lambda_K$, we have*

$$\begin{aligned}
\pi_{1s}(R)(X, Y, Z, U) &= \frac{1}{2} \sum_{A=I,J,K} \gamma_A \otimes \omega_A(X, Y, Z, U) \\
&\quad + \frac{1}{4n} \sum_{A=I,J,K} \langle \xi_X e_i, \xi_Y A e_i \rangle \omega_A(Z, U),
\end{aligned} \tag{6.4}$$

$$\text{Ric}_A^*(X, Y) = -n\gamma_A(X, AY) - \langle \xi_X e_i, \xi_{AY} A e_i \rangle, \tag{6.5}$$

$$\text{Ric}^q(X, Y) = -n \sum_{A=I,J,K} \gamma_A(X, AY) - \sum_{A=I,J,K} \langle \xi_X e_i, \xi_{AY} A e_i \rangle. \tag{6.6}$$

Proof. In equation (6.3), we consider $Z = Ke_i$ and $U = e_i$. Therefore, we obtain

$$4\langle R(X, Y, \cdot, \cdot), \omega_K \rangle = 2R(X, Y, Ke_i, e_i) = 4n\gamma_K(X, Y) - 4\langle \xi_X \xi_Y Ke_i, e_i \rangle.$$

Since $2\text{Ric}_K^*(X, KY) = R(X, Y, Ke_i, e_i)$, the equations of the Lemma follow. \square

A third projection is the map $\pi_1: \Lambda^2 T^* M \otimes \Lambda^2 T^* M \rightarrow \Lambda^2 T^* M \otimes \Lambda_0^2 ES^2 H$ defined by $\pi_1 = \pi_{1es} - \pi_{1s}$. For the curvature tensor $R \in \mathcal{R}$, we have

$$\begin{aligned} \pi_1(R)(X, Y, Z, U) = & \langle \tilde{\mathbf{a}}(\tilde{\nabla}\xi)_{X,Y}Z, U \rangle - \frac{3}{4} \langle \tilde{\mathbf{a}}(\xi \circ \xi)_{X,Y}Z, U \rangle \\ & - \frac{1}{4} \sum_{A=I,J,K} \langle A\tilde{\mathbf{a}}(\xi \circ \xi)_{X,Y}AZ, U \rangle + \langle \tilde{\mathbf{b}}(\xi \otimes \xi)_{X,Y}Z, U \rangle \\ & - \frac{1}{4n} \sum_{A=I,J,K} \langle \xi_X e_i, \xi_Y A e_i \rangle \omega_A(Z, U). \end{aligned} \quad (6.7)$$

Let \mathcal{QK} be the subspace of \mathcal{R} such that $\mathcal{QK} = \mathcal{R} \cap \ker \pi_1$. The space \mathcal{QK} can be seen as the space of possible curvature tensors of a quaternionic Kähler manifold. On $\Lambda^2 T^* M \otimes \Lambda^2 T^* M$, we will consider the extension of the metric $g = \langle \cdot, \cdot \rangle$ defined by

$$\langle a, b \rangle = a(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4})b(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}), \quad (6.8)$$

and write \mathcal{QK}^\perp for the orthogonal complement of \mathcal{QK} in \mathcal{R} , i.e., $\mathcal{R} = \mathcal{QK} + \mathcal{QK}^\perp$. There exists an $Sp(n) Sp(1)$ -map

$$\pi_2: \Lambda^2 T^* M \otimes \Lambda_0^2 ES^2 H \rightarrow \mathcal{QK}^\perp$$

such that the restriction of $\pi^\perp = \pi_2 \circ \pi_1$ to \mathcal{R} is the orthogonal projection $\mathcal{R} \rightarrow \mathcal{QK}^\perp$ and the restriction of π_2 to the orthogonal complement of $\pi_1(\mathcal{R})$ is zero. Therefore, making use of the $Sp(n) Sp(1)$ -map π_2 , we have the following consequence of equation (6.7).

Proposition 6.3. *On an almost quaternion-Hermitian manifold, the components of $\pi^\perp(R)$ in \mathcal{QK}^\perp are linear functions of the components of $\tilde{\nabla}\xi$ and $\xi \otimes \xi$, where $\tilde{\nabla} = \nabla + \xi$ is the minimal $Sp(n) Sp(1)$ -connection.* \square

Since there are components of R in \mathcal{QK} and \mathcal{QK}^\perp which only depend on the Ricci and the q-Ricci tensors, a detailed study of these tensors will refine the above result.

Lemma 6.4. *On an almost quaternion-Hermitian manifold, the Ricci and*

q-Ricci curvature tensors satisfy the identities

$$\begin{aligned} & 3 \operatorname{Ric}(X, Y) - \operatorname{Ric}^q(X, Y) \\ &= \sum_{A=I, J, K} (-2\gamma_A(X, AY) + \langle \xi_X e_i, \xi_{Ae_i} AY \rangle + \langle \xi_X AY, \xi_{e_i} Ae_i \rangle) \end{aligned} \quad (6.9)$$

$$\begin{aligned} & + 4\langle (\tilde{\nabla}_X \xi)_{e_i} Y, e_i \rangle - 4\langle (\tilde{\nabla}_{e_i} \xi)_X Y, e_i \rangle - \langle \xi_X e_i, \xi_{e_i} Y \rangle \\ & - 3\langle \xi_X Y, \xi_{e_i} e_i \rangle - 4\langle \xi_{\xi_{e_i} X} Y, e_i \rangle, \\ 3 \operatorname{Ric} &= \sum_{A=I, J, K} (-(n+2)\gamma_A(\cdot, A\cdot) - \langle \xi \cdot e_i, \xi_A Ae_i \rangle + \langle \xi \cdot e_i \xi_{Ae_i} A\cdot \rangle) \\ & + \sum_{A=I, J, K} \langle \xi \cdot A\cdot, \xi_{e_i} Ae_i \rangle + 4\langle (\tilde{\nabla} \cdot \xi)_{e_i} \cdot, e_i \rangle - 4\langle (\tilde{\nabla}_{e_i} \xi) \cdot \cdot, e_i \rangle \\ & - \langle \xi \cdot e_i, \xi_{e_i} \cdot \rangle - 3\langle \xi \cdot \cdot, \xi_{e_i} e_i \rangle - 4\langle \xi_{\xi_{e_i} \cdot} \cdot, e_i \rangle. \end{aligned} \quad (6.10)$$

Proof. If we take $Y = U = e_i$ and write Y instead of Z in the equation of Lemma 6.1, we will obtain equation (6.9). On the other hand, equation (6.10) is a direct consequence of equations (6.6) and (6.9). \square

The next Lemma contains an algebraic result that we will need to analyse the curvature tensor of a quaternionic Kähler manifold.

Lemma 6.5. *Let $(\mathcal{V}, I, J, K, \langle \cdot, \cdot \rangle)$ be a quaternionic vector space of dimension $4n$, $n > 1$. If γ_I , γ_J and γ_K are three two-forms such that*

$$\gamma_I \wedge \omega_J = \gamma_J \wedge \omega_I, \quad \gamma_J \wedge \omega_K = \gamma_K \wedge \omega_J, \quad \gamma_K \wedge \omega_I = \gamma_I \wedge \omega_K,$$

then $\gamma_A = c \omega_A$, for $A = I, J, K$, where

$$2n c = \langle \gamma_I, \omega_I \rangle = \langle \gamma_J, \omega_J \rangle = \langle \gamma_K, \omega_K \rangle.$$

Proof. If we compute the contractions

$$\begin{aligned} (\gamma_I \wedge \omega_J)(X, Y, Ie_i, e_i) &= (\gamma_J \wedge \omega_I)(X, Y, Ie_i, e_i), \\ (\gamma_I \wedge \omega_J)(X, Y, Je_i, e_i) &= (\gamma_J \wedge \omega_I)(X, Y, Je_i, e_i), \end{aligned}$$

we will get

$$\begin{aligned} & 2(n-1)\gamma_J(X, Y) + \langle \gamma_J, \omega_I \rangle \omega_I(X, Y) \\ &= \gamma_I(X, KY) + \gamma_I(KX, Y) + \langle \gamma_I, \omega_I \rangle \omega_J(X, Y), \\ & 2(n-1)\gamma_I(X, Y) + \langle \gamma_I, \omega_J \rangle \omega_J(X, Y) \\ &= -\gamma_J(X, KY) - \gamma_J(KX, Y) + \langle \gamma_J, \omega_J \rangle \omega_I(X, Y). \end{aligned}$$

As a consequence of these equations, $K\gamma_I = -\gamma_I$ and $K\gamma_J = -\gamma_J$, i.e., γ_I and γ_J are anti-Hermitian for K . Moreover, we get the following identities

$$\begin{aligned} 2(n-1)\gamma_I + \langle \gamma_I, \omega_J \rangle \omega_J &= 2K_{(1)}\gamma_J + \langle \gamma_J, \omega_J \rangle \omega_I, \\ 2(n-1)\gamma_J + \langle \gamma_J, \omega_I \rangle \omega_I &= -2K_{(1)}\gamma_I + \langle \gamma_I, \omega_I \rangle \omega_J. \end{aligned}$$

By similar arguments, cyclically permuting I, J, K , we obtain that γ_J and γ_K are anti-Hermitian for I , γ_K and γ_I are anti-Hermitian for J , and

$$2(n-1)\gamma_I = 2K_{(1)}\gamma_J + \langle \gamma_J, \omega_J \rangle \omega_I = -2J_{(1)}\gamma_K + \langle \gamma_K, \omega_K \rangle \omega_I, \quad (6.11)$$

$$2(n-1)\gamma_J = -2K_{(1)}\gamma_I + \langle \gamma_I, \omega_I \rangle \omega_J = 2I_{(1)}\gamma_K + \langle \gamma_K, \omega_K \rangle \omega_J, \quad (6.12)$$

$$2(n-1)\gamma_K = -2I_{(1)}\gamma_J + \langle \gamma_J, \omega_J \rangle \omega_K = 2J_{(1)}\gamma_I + \langle \gamma_I, \omega_I \rangle \omega_K. \quad (6.13)$$

From equations (6.13), taking equations (6.11) into account, we have

$$2(n-1)J_{(1)}\gamma_K = 2K_{(1)}\gamma_J + \langle \gamma_J, \omega_J \rangle \omega_I = -2\gamma_I + \langle \gamma_I, \omega_I \rangle \omega_I = 2(n-1)\gamma_I.$$

Therefore

$$\gamma_I = J_{(1)}\gamma_K = \frac{1}{2n}\langle \gamma_I, \omega_I \rangle \omega_I.$$

Since by an analogous argument we also have $2n\gamma_K = \langle \gamma_K, \omega_K \rangle \omega_K$, we find $2n\gamma_I = 2J_{(1)}\gamma_K = \langle \gamma_K, \omega_K \rangle \omega_I$. Thus $\langle \gamma_K, \omega_K \rangle = \langle \gamma_I, \omega_I \rangle$. \square

Now we give an alternative proof of the already classical result that any quaternionic Kähler manifold is Einstein [2, 9, 14]. In our view, in the proof we present here, the rôle played by the $Sp(n)Sp(1)$ -structure is seen in a more natural way. Likewise, we also provide alternative proofs for some known additional information about quaternionic Kähler manifolds [18, 8].

Theorem 6.6. *A quaternionic Kähler $4n$ -manifold M , $n > 1$, is Einstein, q -Einstein and locally Ric_A^* -Einstein for $A = I, J, K$. Moreover, if R is the curvature of M , then*

- (i) $\text{Ric} = (n+2)cg$, $\text{Ric}_A^* = ncg$ and $\text{Ric}^q = 3ncg$, where $2nc = \langle \gamma_I, \omega_I \rangle = \langle \gamma_J, \omega_J \rangle = \langle \gamma_K, \omega_K \rangle$, and $\gamma_I = d\lambda_I + \lambda_J \wedge \lambda_K$;
- (ii) $\pi_{\mathbb{R}_a + \mathbb{R}_b}(R) = \frac{c}{8}(\pi_2 + 2\pi_1)$, where $\pi_{\mathbb{R}_a + \mathbb{R}_b}$ is the projection $\mathcal{R} \rightarrow \mathbb{R}_a + \mathbb{R}_b$;
- (iii) $R \in S^4E + \mathbb{R}(\pi_2 + 2\pi_1) = \mathcal{QK}$.

Proof. Since the manifold is quaternionic Kähler, we have

$$\begin{aligned} d\omega_I &= \lambda_K \wedge \omega_J - \lambda_J \wedge \omega_K, \\ d\omega_J &= \lambda_I \wedge \omega_K - \lambda_K \wedge \omega_I, \\ d\omega_K &= \lambda_J \wedge \omega_I - \lambda_I \wedge \omega_J. \end{aligned}$$

Now, writing $\gamma_I = d\lambda_I + \lambda_J \wedge \lambda_K$, from $d^2\omega_I = d^2\omega_J = d^2\omega_K = 0$, we obtain

$$\gamma_K \wedge \omega_J = \gamma_J \wedge \omega_K, \quad \gamma_I \wedge \omega_K = \gamma_K \wedge \omega_I, \quad \gamma_J \wedge \omega_I = \gamma_I \wedge \omega_J.$$

Finally, using Lemma 6.5 and equations (6.5), (6.6) and (6.10), it follows that M is Einstein and we have part (i).

For part (ii), taking Propositions 4.2 and 4.3 into account and using part (i), we have

$$\begin{aligned} \text{Ric}(\pi_{\mathbb{R}_a}(R)) &= \frac{1}{2}(\text{Ric}(R) + \text{Ric}^q(R)) = (2n+1)c\,g, \\ \text{Ric}(\pi_{\mathbb{R}_b}(R)) &= \frac{1}{2}(\text{Ric}(R) - \text{Ric}^q(R)) = -(n-1)c\,g. \end{aligned}$$

Now, taking Proposition 4.2(iii) and Proposition 4.3(iii) into account, we obtain

$$\pi_{\mathbb{R}_a}(R) = \frac{c}{12}(\pi_2 + 6\pi_1), \quad \pi_{\mathbb{R}_b}(R) = \frac{c}{24}(\pi_2 - 6\pi_1).$$

Hence part (ii) follows.

Finally, writing $R_1 = R - \pi_{\mathbb{R}_a + \mathbb{R}_b}(R)$, using equation (6.3), we have

$$R_1(X, Y, Z, U) - R_1(X, Y, AZ, AU) = 0,$$

for $A = I, J, K$. From this identity it is not hard to check $L_\sigma(R_1) = 12R_1$. Then, by Proposition 3.2, $R_1 \in S^4E$ and we have part (iii). \square

At this point, we can be a little more precise about the space \mathcal{QK}^\perp .

Proposition 6.7. *For an almost quaternion-Hermitian manifold, if we denote $\mathbb{R}_{\mathcal{QK}} = \mathbb{R}(\pi_2 + 2\pi_1)$, then*

- (i) *the orthogonal complement $\mathbb{R}_{\mathcal{QK}}^\perp$ of $\mathbb{R}_{\mathcal{QK}}$ in $\mathbb{R}_a + \mathbb{R}_b$ is given by $\mathbb{R}_{\mathcal{QK}^\perp} = \mathbb{R}((n+2)\pi_2 - 18n\pi_1)$;*
- (ii) *the space \mathcal{QK}^\perp decomposes into irreducible $Sp(n)Sp(1)$ -modules as*

$$\begin{aligned} \mathcal{QK}^\perp &= \mathbb{R}_{\mathcal{QK}^\perp} + V^{22} + (\Lambda_0^2 E)_a + \Lambda_0^4 E + (\Lambda_0^2 E)_b + V^{31} S^2 H \\ &\quad + (S^2 E S^2 H)_a + V^{211} S^2 H + (S^2 E S^2 H)_b + \Lambda_0^2 E S^2 H \\ &\quad + V^{22} S^4 H + \Lambda_0^2 E S^4 H + S^4 H; \end{aligned}$$

- (iii) *the component of R in $\mathbb{R}_{\mathcal{QK}}$ is determined by $\text{Ric}(\pi_{\mathcal{QK}}(R))$ which is given by*

$$\text{Ric}(\pi_{\mathcal{QK}}(R)) = \frac{n+2}{2(5n+1)}(\pi_{\mathbb{R}}(\text{Ric}) + 3\pi_{\mathbb{R}}(\text{Ric}^q));$$

(iv) if we denote $\text{Ric}_{\mathcal{QK}^\perp} = \text{Ric}(\pi_{\mathcal{QK}^\perp}(R))$, the component of R in $\mathbb{R}_{\mathcal{QK}^\perp}$ is determined by $\pi_{\mathbb{R}}(\text{Ric}_{\mathcal{QK}^\perp})$ which is given by

$$\pi_{\mathbb{R}}(\text{Ric}_{\mathcal{QK}^\perp}) = \frac{9n}{2(5n+1)}(\pi_{\mathbb{R}}(\text{Ric}) - \frac{n+2}{3n}\pi_{\mathbb{R}}(\text{Ric}^q)).$$

Proof. Part (i) follows by straightforward computations. In such computations we will obtain

$$\langle \pi_2, \pi_2 \rangle = 36\langle \pi_1, \pi_1 \rangle = 288n(4n-1), \quad \langle \pi_1, \pi_2 \rangle = 144n.$$

We recall that the scalar product for these tensors is given by equation (6.8). Part (ii) is a direct consequence of part (i) and results contained in some previous Sections. For parts (ii) and (iii), note that $\text{Ric}_{\mathcal{QK}}$ and $\pi_{\mathbb{R}}(\text{Ric}_{\mathcal{QK}^\perp})$ are the Ricci curvatures which respectively correspond to the components of the curvature in $\mathbb{R}(\pi_2 + 2\pi_1)$ and $\mathbb{R}((n+2)\pi_2 - 18n\pi_1)$. Also for the q-Ricci curvatures we have

$$\begin{aligned} \text{Ric}^q(\pi_{\mathcal{QK}}(R)) &= \frac{3n}{2(5n+1)}(\pi_{\mathbb{R}}(\text{Ric}) + 3\pi_{\mathbb{R}}(\text{Ric}^q)), \\ \pi_{\mathbb{R}}(\text{Ric}_{\mathcal{QK}^\perp}^q) &= -\frac{3n}{2(5n+1)}(\pi_{\mathbb{R}}(\text{Ric}) - \frac{n+2}{3n}\pi_{\mathbb{R}}(\text{Ric}^q)), \end{aligned}$$

where $\text{Ric}_{\mathcal{QK}^\perp}^q = \text{Ric}^q(\pi_{\mathcal{QK}^\perp}(R))$. □

Now our purpose is to derive some further consequences of the identities $d^2\omega_I = d^2\omega_J = d^2\omega_K = 0$. From equation (6.2), we have

$$\begin{aligned} d\omega_I(X, Y, Z) &= (\lambda_K \wedge \omega_J - \lambda_J \wedge \omega_K)(X, Y, Z) \\ &\quad + \sum_{X, Y, Z} (\langle Y, I\xi_X Z \rangle - \langle Y, \xi_X IZ \rangle). \end{aligned}$$

Now, since $d^2\omega_I = 0$ and $\tilde{\nabla} = \nabla + \xi$, it is not hard to obtain

$$\begin{aligned}
0 = & (\gamma_K \wedge \omega_J - \gamma_J \wedge \omega_K)(X, Y, Z, U) \\
& + \mathfrak{S}_{Y,Z,U} (\langle Z, I(\tilde{\nabla}_X \xi)_Y U \rangle - \langle Z, (\tilde{\nabla}_X \xi)_Y IU, \rangle) \\
& - \mathfrak{S}_{X,Z,U} (\langle Z, I(\tilde{\nabla}_Y \xi)_X U \rangle - \langle Z, (\tilde{\nabla}_Y \xi)_X IU, \rangle) \\
& + \mathfrak{S}_{X,Y,U} (\langle Y, I(\tilde{\nabla}_Z \xi)_X U \rangle - \langle Y, (\tilde{\nabla}_Z \xi)_X IU \rangle) \\
& - \mathfrak{S}_{X,Y,Z} (\langle Y, I(\tilde{\nabla}_U \xi)_X Z \rangle - \langle Y, (\tilde{\nabla}_U \xi)_X IZ, \rangle) \\
& - \mathfrak{S}_{Y,Z,U} (\langle Z, (\xi_X I \xi)_Y U \rangle - \langle Z, (\xi_X \xi I)_Y U \rangle) \\
& + \mathfrak{S}_{X,Z,U} (\langle Z, (\xi_Y I \xi)_X U \rangle - \langle Z, (\xi_Y \xi I)_X U, \rangle) \\
& - \mathfrak{S}_{X,Y,U} (\langle Y, (\xi_Z I \xi)_X U \rangle - \langle Y, (\xi_Z \xi I)_X U \rangle) \\
& + \mathfrak{S}_{X,Y,Z} (\langle Y, (\xi_U I \xi)_X Z \rangle - \langle Y, (\xi_U \xi I)_X Z, \rangle).
\end{aligned} \tag{6.14}$$

If we respectively replace X, Y, Z, U by X, JY, Ke_i, e_i in last identity, proceed in an analogous way for $d^2\omega_J$ and $d^2\omega_K$, and finally summing the obtained expressions, we get

$$\begin{aligned}
0 = & - \sum_{A=I,J,K} ((2n-1)\gamma_A(X, AY) - \langle \gamma_A, \omega_A \rangle \langle X, Y \rangle) \\
& + \mathfrak{S}_{IJK} (\langle \gamma_J, \omega_K \rangle \omega_I(X, Y) - \gamma_I(JX, KY)) \\
& + \sum_{A=I,J,K} (-2\langle \xi_X e_i, \xi_{AY} A e_i \rangle - \langle \xi_{e_i} X, A \xi_{AY} e_i \rangle + \langle \xi_{e_i} X, \xi_{AY} A e_i \rangle \\
& \quad + \langle X, \xi_{AY} \xi_{e_i} A e_i \rangle + \langle X, \xi_{\xi_{e_i} A e_i} AY \rangle + \langle \xi_{e_i} X, \xi_{A e_i} AY \rangle \\
& \quad + \langle X, \xi_{\xi_{e_i} AY} A e_i \rangle + \langle X, (\tilde{\nabla}_{e_i} \xi)_{A e_i} AY \rangle \\
& \quad + \langle X, (\tilde{\nabla}_{e_i} \xi)_{AY} A e_i \rangle - \langle X, (\tilde{\nabla}_{AY} \xi)_{e_i} A e_i \rangle) \\
& + \mathfrak{S}_{IJK} (\langle \xi_{e_i} IX, K \xi_{JY} e_i \rangle - \langle \xi_{e_i} IX, \xi_{JY} K e_i \rangle - \langle IX, \xi_{JY} \xi_{e_i} K e_i \rangle \\
& \quad - \langle X, I \xi_{\xi_{e_i} K e_i} JY \rangle + \langle \xi_{e_i} IX, \xi_{K e_i} JY, \rangle + \langle X, I \xi_{\xi_{e_i} JY} K e_i \rangle \\
& \quad - \langle X, I(\tilde{\nabla}_{JY} \xi)_{e_i} K e_i \rangle + \langle X, I(\tilde{\nabla}_{e_i} \xi)_{JY} K e_i \rangle \\
& \quad - \langle X, I(\tilde{\nabla}_{e_i} \xi)_{K e_i} JY \rangle).
\end{aligned} \tag{6.15}$$

Now by computing the $\Lambda_0^2 E$ -components of the bilinear forms contained in this identity we get

$$\begin{aligned} n\pi_{\Lambda_0^2 E}(\sum_{A=I,J,K} \gamma_A(\cdot, A\cdot)) \\ = - \sum_{A=I,J,K} \pi_{\Lambda_0^2 E}(\langle \xi \cdot e_i, \xi_A A e_i \rangle - \langle \xi_{e_i} \cdot, \xi_{A e_i} A \cdot \rangle) \\ + \sum_{A=I,J,K} \pi_{S^2 T^*}(\langle \cdot, \xi_{\xi_{e_i} A e_i} A \cdot \rangle + \langle \cdot, (\tilde{\nabla}_{e_i} \xi)_{A e_i} A \cdot \rangle). \end{aligned} \quad (6.16)$$

Note that if we compute the corresponding \mathbb{R} -components, we will obtain $\pi_{\mathbb{R}}(\sum_{A=I,J,K} \gamma_A(\cdot, A\cdot)) = -1/2n \sum_{A=I,J,K} \langle \gamma_A, \omega_A \rangle \langle \cdot, \cdot \rangle$ as it is expected. Finally, computing the $S^2 E S^2 H$ -components in equation (6.15), we obtain

$$\begin{aligned} -2(n-1)\pi_{S^2 E S^2 H}(\sum_{A=I,J,K} \gamma_A(\cdot, A\cdot)) \\ = 2\pi_{S^2 E S^2 H}(\sum_{A=I,J,K} \langle \xi \cdot e_i, \xi_A A e_i \rangle) \\ - \pi_{S^2 T^*}(\sum_{A=I,J,K} \langle \xi_{e_i} \cdot, \xi_A A e_i \rangle + \mathfrak{S}_{IJK} \langle \xi_{e_i} I \cdot, \xi_K J e_i \rangle) \\ + \pi_{S^2 T^*}(\sum_{A=I,J,K} \langle \xi_A \cdot, \xi_{e_i} A e_i \rangle + \mathfrak{S}_{IJK} \langle \xi_I J \cdot, \xi_{e_i} K e_i \rangle) \\ + \pi_{S^2 T^*}(\sum_{A=I,J,K} \langle \xi_{e_i} \cdot, A \xi_A e_i \rangle + \mathfrak{S}_{IJK} \langle \xi_{e_i} I \cdot, J \xi_K e_i \rangle) \\ + \pi_{S^2 T^*}(\mathfrak{S}_{IJK} \langle \cdot, I \xi_{\xi_{e_i} K} J e_i \rangle - \sum_{A=I,J,K} \langle \cdot, \xi_{\xi_{e_i} A} A e_i \rangle) \\ + \pi_{S^2 T^*}(\sum_{A=I,J,K} \langle \cdot, (\tilde{\nabla}_A \xi)_{e_i} A e_i \rangle - \mathfrak{S}_{IJK} \langle \cdot, I(\tilde{\nabla}_K \xi)_{e_i} J e_i \rangle) \\ - \pi_{S^2 T^*}(\sum_{A=I,J,K} \langle \cdot, (\tilde{\nabla}_{e_i} \xi)_A A e_i \rangle - \mathfrak{S}_{IJK} \langle \cdot, I(\tilde{\nabla}_{e_i} \xi)_K J e_i \rangle). \end{aligned} \quad (6.17)$$

At this point we can give a more detailed description for the components of the Ricci curvature tensors. In fact, from equations (6.6) and (6.10), using the identity (6.16), we get

$$\pi_{\mathbb{R}}(\text{Ric}^q) = \frac{1}{2} \sum_{A=I,J,K} (\langle \gamma_A, \omega_A \rangle - \frac{1}{2n} \langle \xi_{e_i} e_j, \xi_{A e_i} A e_j \rangle) \langle \cdot, \cdot \rangle, \quad (6.18)$$

$$\begin{aligned}
12n\pi_{\mathbb{R}}(\text{Ric}) = & (-3\langle \xi_{e_i}e_i, \xi_{e_j}e_j \rangle - 5\langle \xi_{e_i}e_j, \xi_{e_j}e_i \rangle + 8\langle (\tilde{\nabla}_{e_i}\xi)_{e_j}e_i, e_j \rangle \\
& + \sum_{A=I,J,K} (2(n+2)\langle \gamma_A, \omega_A \rangle + \langle \xi_{e_i}Ae_i, \xi_{e_j}Ae_j \rangle \\
& + \langle \xi_{e_i}e_j, \xi_{Ae_j}Ae_i \rangle - \langle \xi_{e_i}e_j, \xi_{Ae_i}Ae_j \rangle) \langle \cdot, \cdot \rangle.
\end{aligned} \tag{6.19}$$

Taking traces gives:

Proposition 6.8. *The scalar curvature and q -scalar curvature are*

$$\begin{aligned}
\text{scal} = & \frac{2(n+2)}{3} \sum_A \langle \gamma_A, \omega_A \rangle + \frac{7}{3} \|\xi_{33}\|^2 - \frac{1}{3} \|\xi_{K3}\|^2 + \frac{2n^2+3n+2}{3n} \|\xi_{E3}\|^2 \\
& - \frac{1}{3} \|\xi_{3H}\|^2 - \frac{7}{3} \|\xi_{KH}\|^2 + \frac{2(4n^2+6n+1)}{3n} \|\xi_{EH}\|^2 \\
& - \frac{16(2n+1)(n+1)}{n} d^*\theta^\xi,
\end{aligned} \tag{6.20}$$

and

$$\begin{aligned}
\text{scal}^q = & 2n \sum_A \langle \gamma_A, \omega_A \rangle + \|\xi_{33}\|^2 + \|\xi_{K3}\|^2 + \|\xi_{E3}\|^2 \\
& - 2\|\xi_{3H}\|^2 - 9\|\xi_{KH}\|^2 - \frac{2}{3}\|\xi_{EH}\|^2.
\end{aligned} \tag{6.21}$$

□

Computing the $\Lambda_0^2 E$ -components gives

$$\begin{aligned}
\pi_{\Lambda_0^2 E}(\text{Ric}^q) = & - \sum_{A=I,J,K} \pi_{S^2 T^*}(\langle \cdot, \xi_{\xi_{e_i}Ae_i}A \cdot \rangle + \langle \cdot, (\tilde{\nabla}_{e_i}\xi)_{Ae_i}A \cdot \rangle) \\
& - \sum_{A=I,J,K} \pi_{\Lambda_0^2 E}(\langle \xi_{e_i} \cdot, \xi_{Ae_i}A \cdot \rangle),
\end{aligned} \tag{6.22}$$

$$\begin{aligned}
3\pi_{\Lambda_0^2 E}(\text{Ric}) = & -\pi_{\Lambda_0^2 E}(\langle \xi_{e_i} \cdot, \xi_{e_i} \cdot \rangle + 3\langle \xi \cdot, \xi_{e_i}e_i \rangle + 4\langle \xi_{\xi_{e_i} \cdot} \cdot, e_i \rangle) \\
& - \frac{n+2}{n} \sum_{A=I,J,K} \pi_{S^2 T^*}(\langle \cdot, \xi_{\xi_{e_i}Ae_i}A \cdot \rangle + \langle \cdot, (\tilde{\nabla}_{e_i}\xi)_{Ae_i}A \cdot \rangle) \\
& + \sum_{A=I,J,K} \pi_{\Lambda_0^2 E}(\langle \xi \cdot A \cdot, \xi_{e_i}Ae_i \rangle + \langle \xi_{e_i} \cdot, \xi_{Ae_i}A \cdot \rangle) \\
& + \sum_{A=I,J,K} \pi_{\Lambda_0^2 E}(\frac{2}{n} \langle \xi_{e_i} \cdot, \xi_{A \cdot}Ae_i \rangle - \frac{n+2}{n} \langle \xi_{e_i} \cdot, \xi_{Ae_i}A \cdot \rangle) \\
& + 4\pi_{\Lambda_0^2 E}(\langle (\tilde{\nabla} \cdot \xi)_{e_i} \cdot, e_i \rangle - \langle (\tilde{\nabla}_{e_i} \xi) \cdot, e_i \rangle).
\end{aligned} \tag{6.23}$$

Explicit expressions for the S^2ES^2H -components $\pi_{S^2ES^2H}(\text{Ric}^q)$ of Ric^q and $\pi_{S^2ES^2H}(\text{Ric})$ of Ric can be easily obtained from equations (6.6), (6.10) and (6.17). Because of their sizes, we will not write such expressions, but it is clear that such expressions depend linearly on $\xi \otimes \xi$ and $\tilde{\nabla}\xi$.

It remains to analyse the $\Lambda_0^2ES^2H$ -component of Ric^q . For this purpose, replace Z and U by Ie_i and e_i in equation (6.14), perform analogous operations for J and K and then add the expressions obtained to get

$$\begin{aligned}
0 = & 2 \sum_{A=I,J,K} (\gamma_A(X, AY) + \gamma_A(AX, Y)) \\
& + \sum_{IJK} (\langle \gamma_J, \omega_K \rangle - \langle \gamma_K, \omega_J \rangle) \omega_I(X, Y) + \langle \xi_X e_i, \xi_{e_i} Y \rangle - \langle \xi_Y e_i, \xi_{e_i} X \rangle \\
& + 3 \langle \xi_X Y, \xi_{e_i} e_i \rangle - 3 \langle \xi_Y X, \xi_{e_i} e_i \rangle + 4 \langle X, \xi_{\xi_{e_i} Y} e_i \rangle - 4 \langle Y, \xi_{\xi_{e_i} X} e_i \rangle \\
& + \sum_{A=I,J,K} (- \langle \xi_X e_i, \xi_{Ae_i} AY \rangle + \langle \xi_Y e_i, \xi_{Ae_i} AX \rangle + \langle X, \xi_{\xi_{e_i} Ae_i} AY \rangle \\
& \quad - \langle Y, \xi_{\xi_{e_i} Ae_i} AX \rangle - \langle \xi_X AY, \xi_{e_i} Ae_i \rangle + \langle \xi_Y AX, \xi_{e_i} Ae_i \rangle \\
& \quad + \langle \xi_{e_i} X, \xi_{Ae_i} AY \rangle - \langle \xi_{e_i} Y, \xi_{Ae_i} AX \rangle + \langle X, (\tilde{\nabla}_{e_i} \xi)_{Ae_i} AY \rangle \\
& \quad - \langle Y, (\tilde{\nabla}_{e_i} \xi)_{Ae_i} AX \rangle) \\
& - 4 \langle X, (\tilde{\nabla}_Y \xi)_{e_i} e_i \rangle + 4 \langle Y, (\tilde{\nabla}_X \xi)_{e_i} e_i \rangle + 4 \langle X, (\tilde{\nabla}_{e_i} \xi)_Y e_i \rangle - 4 \langle Y, (\tilde{\nabla}_{e_i} \xi)_X e_i \rangle.
\end{aligned}$$

From this last identity it is straightforward to derive the $\Lambda_0^2ES^2H$ -component of $\sum_{A=I,J,K} \gamma_A(\cdot, A\cdot)$ which is given by

$$\begin{aligned}
& 2\pi_{\Lambda_0^2ES^2H} \left(\sum_{A=I,J,K} \gamma_A(X, AY) \right) \\
& = -\pi_{\Lambda_0^2ES^2H} (\langle \xi_X e_i, \xi_{e_i} Y \rangle + 3 \langle \xi_X Y, \xi_{e_i} e_i \rangle) \\
& \quad - 4\pi_{\Lambda_0^2ES^2H} (\langle X, \xi_{\xi_{e_i} Y} e_i \rangle + \langle X, (\tilde{\nabla}_{e_i} \xi)_Y e_i \rangle - \langle X, (\tilde{\nabla}_Y \xi)_{e_i} e_i \rangle) \\
& \quad + \sum_{A=I,J,K} \pi_{\Lambda_0^2ES^2H} (\langle \xi_X e_i, \xi_{Ae_i} AY \rangle + \langle \xi_X AY, \xi_{e_i} Ae_i \rangle - \langle \xi_{e_i} X, \xi_{Ae_i} AY \rangle) \\
& \quad - \sum_{A=I,J,K} \pi_{\Lambda^2T^*} (\langle X, \xi_{\xi_{e_i} Ae_i} AY \rangle + \langle X, (\tilde{\nabla}_{e_i} \xi)_{Ae_i} AY \rangle).
\end{aligned}$$

Therefore, using this identity and equation (6.6), we deduce the $\Lambda_0^2ES^2H$ -

component of the q-Ricci tensor which is given by

$$\begin{aligned}
& \frac{2}{n} \pi_{\Lambda_0^2 ES^2 H}(\text{Ric}^q)(X, Y) \\
&= \pi_{\Lambda_0^2 ES^2 H}(\langle \xi_X e_i, \xi_{e_i} Y \rangle + 3\langle \xi_X Y, \xi_{e_i} e_i \rangle) \\
&\quad + 4\pi_{\Lambda_0^2 ES^2 H}(\langle X, \xi_{\xi_{e_i} Y} e_i \rangle + \langle X, (\tilde{\nabla}_{e_i} \xi)_Y e_i \rangle - \langle X, (\tilde{\nabla}_Y \xi)_{e_i} e_i \rangle) \\
&\quad - \sum_{A=I, J, K} \pi_{\Lambda_0^2 ES^2 H}(\langle \xi_X e_i, \xi_{Ae_i} AY \rangle + \langle \xi_X AY, \xi_{e_i} Ae_i \rangle - \langle \xi_{e_i} X, \xi_{Ae_i} AY \rangle) \\
&\quad + \sum_{A=I, J, K} \pi_{\Lambda^2 T^*}(\langle X, \xi_{\xi_{e_i} Ae_i} AY \rangle + \langle X, (\tilde{\nabla}_{e_i} \xi)_{Ae_i} AY \rangle) \\
&\quad - \frac{2}{n} \sum_{A=I, J, K} \pi_{\Lambda_0^2 ES^2 H}(\langle \xi_X e_i, \xi_{AY} Ae_i \rangle).
\end{aligned} \tag{6.24}$$

Equations (6.18)–(6.24) and the above description of the $S^2 ES^2 H$ -components of Ric^q and Ric give rise to the following result. Here we will follow the notation used in §5 writing the components of the intrinsic torsion ξ as ξ_{UF} , for $U = 3, K, E$ and $F = 3, H$.

Theorem 6.9. *Let M be an almost quaternion-Hermitian $4n$ -manifold, $n > 1$, with minimal $\text{Sp}(n) \text{Sp}(1)$ -connection $\tilde{\nabla} = \nabla + \xi$. The tensors $\sum_A \langle \gamma_A, \omega_A \rangle$, $\tilde{\nabla} \xi_{UF}$ and $\xi_{UF} \odot \xi_{VG}$ contribute to the components of the q-Ricci curvature Ric^q via equation (6.6) and to the Ricci curvature Ric via equation (6.10) if and only if there is a tick in the corresponding place in Table 6.1. \square*

Taking Proposition 6.7 (iii) and (iv) into account, using equations (6.18) and (6.19) we have the following expressions which determine the curvature components in $\mathbb{R}_{\mathcal{QK}}$ and $\mathbb{R}_{\mathcal{QK}^\perp}$, respectively,

$$\begin{aligned}
\frac{24n(5n+1)}{n+2} \text{Ric}_{\mathcal{QK}} &= (-3\langle \xi_{e_i} e_i, \xi_{e_j} e_j \rangle - 5\langle \xi_{e_i} e_j, \xi_{e_j} e_i \rangle + 8\langle (\tilde{\nabla}_{e_i} \xi)_{e_j} e_i, e_j \rangle \\
&\quad + \sum_{A=I, J, K} (4(5n+1)\langle \gamma_A, \omega_A \rangle + \langle \xi_{e_i} Ae_i, \xi_{e_j} Ae_j \rangle) \\
&\quad + \langle \xi_{e_i} e_j, \xi_{Ae_j} Ae_i \rangle - 10\langle \xi_{e_i} e_j, \xi_{Ae_i} Ae_j \rangle) \langle \cdot, \cdot \rangle,
\end{aligned} \tag{6.25}$$

$$\begin{aligned}
\frac{8(5n+1)}{3} \pi_{\mathbb{R}}(\text{Ric}_{\mathcal{QK}^\perp}) &= (-3\langle \xi_{e_i} e_i, \xi_{e_j} e_j \rangle - 5\langle \xi_{e_i} e_j, \xi_{e_j} e_i \rangle + 8\langle (\tilde{\nabla}_{e_i} \xi)_{e_j} e_i, e_j \rangle \\
&\quad + \sum_{A=I, J, K} (\langle \xi_{e_i} Ae_i, \xi_{e_j} Ae_j \rangle + \langle \xi_{e_i} e_j, \xi_{Ae_j} Ae_i \rangle + 2\langle \xi_{e_i} e_j, \xi_{Ae_i} Ae_j \rangle)) \langle \cdot, \cdot \rangle.
\end{aligned} \tag{6.26}$$

	Ric ^q				Ric		
	\mathbb{R}	$\Lambda_0^2 E$	$S^2 ES^2 H$	$\Lambda_0^2 ES^2 H$	\mathbb{R}	$\Lambda_0^2 E$	$S^2 ES^2 H$
$\sum_A \langle \gamma_A, \omega_A \rangle$	✓				✓		
$\tilde{\nabla} \xi_{33}$			✓	✓			✓
$\tilde{\nabla} \xi_{K3}$			✓	✓			✓
$\tilde{\nabla} \xi_{E3}$			✓	✓			✓
$\tilde{\nabla} \xi_{3H}$		✓	✓	✓		✓	✓
$\tilde{\nabla} \xi_{KH}$		✓	✓	✓		✓	✓
$\tilde{\nabla} \xi_{EH}$		✓	✓	✓	✓	✓	✓
$\xi_{33} \otimes \xi_{33}$	✓	✓	✓		✓	✓	✓
$\xi_{K3} \otimes \xi_{K3}$	✓	✓	✓		✓	✓	✓
$\xi_{E3} \otimes \xi_{E3}$	✓	✓	✓		✓	✓	✓
$\xi_{3H} \otimes \xi_{3H}$	✓	✓	✓		✓	✓	✓
$\xi_{KH} \otimes \xi_{KH}$	✓	✓	✓		✓	✓	✓
$\xi_{EH} \otimes \xi_{EH}$	✓	✓	✓		✓	✓	✓
$\xi_{33} \odot \xi_{K3}$		✓	✓	✓		✓	✓
$\xi_{33} \odot \xi_{E3}$		✓		✓		✓	
$\xi_{33} \odot \xi_{3H}$			✓	✓			✓
$\xi_{33} \odot \xi_{KH}$			✓	✓			✓
$\xi_{33} \odot \xi_{EH}$				✓			
$\xi_{K3} \odot \xi_{E3}$		✓	✓	✓		✓	✓
$\xi_{K3} \odot \xi_{3H}$			✓	✓			✓
$\xi_{K3} \odot \xi_{KH}$			✓	✓			✓
$\xi_{K3} \odot \xi_{EH}$			✓	✓			✓
$\xi_{E3} \odot \xi_{3H}$				✓			
$\xi_{E3} \odot \xi_{KH}$			✓	✓			✓
$\xi_{E3} \odot \xi_{EH}$			✓	✓			✓
$\xi_{3H} \odot \xi_{KH}$		✓	✓	✓		✓	✓
$\xi_{3H} \odot \xi_{EH}$		✓		✓		✓	
$\xi_{KH} \odot \xi_{EH}$		✓	✓	✓		✓	✓

Table 6.1: Ricci curvatures from Theorem 6.9.

Due to Proposition 4.2(iv) and Proposition 4.3(iv), the curvature components in $(\Lambda_0^2 E)_a$ and $(\Lambda_0^2 E)_b$ are determined respectively by $2 \operatorname{Ric}_{(\Lambda_0^2 E)_a} = \pi_{\Lambda_0^2 E}(\operatorname{Ric} + \operatorname{Ric}^q)$ and $2 \operatorname{Ric}_{(\Lambda_0^2 E)_b} = \pi_{\Lambda_0^2 E}(\operatorname{Ric} - \operatorname{Ric}^q)$. Using equations (6.22) and (6.23), we obtain the following expressions

$$\begin{aligned}
6 \operatorname{Ric}_{(\Lambda_0^2 E)_a} = & -\pi_{\Lambda_0^2 E}(\langle \xi \cdot e_i, \xi_{e_i} \cdot \rangle + 3\langle \xi \cdot, \xi_{e_i} e_i \rangle + 4\langle \xi_{\xi_{e_i} \cdot}, e_i \rangle) \\
& - \frac{2(2n+1)}{n} \sum_{A=I,J,K} \pi_{S^2 T^*}(\langle \cdot, \xi_{\xi_{e_i} A e_i} A \cdot \rangle + \langle \cdot, (\tilde{\nabla}_{e_i} \xi)_{A e_i} A \cdot \rangle) \\
& + \sum_{A=I,J,K} \pi_{\Lambda_0^2 E}(\langle \xi \cdot A, \xi_{e_i} A e_i \rangle + \langle \xi \cdot e_i, \xi_{A e_i} A \cdot \rangle) \\
& + \sum_{A=I,J,K} \pi_{\Lambda_0^2 E}(\frac{2}{n} \langle \xi \cdot e_i, \xi_A A e_i \rangle - \frac{2(2n+1)}{n} \langle \xi_{e_i} \cdot, \xi_{A e_i} A \cdot \rangle) \\
& + 4\pi_{\Lambda_0^2 E}(\langle (\tilde{\nabla} \cdot \xi)_{e_i} \cdot, e_i \rangle - \langle (\tilde{\nabla}_{e_i} \xi) \cdot, e_i \rangle),
\end{aligned} \tag{6.27}$$

$$\begin{aligned}
6 \operatorname{Ric}_{(\Lambda_0^2 E)_b} = & -\pi_{\Lambda_0^2 E}(\langle \xi \cdot e_i, \xi_{e_i} \cdot \rangle + 3\langle \xi \cdot, \xi_{e_i} e_i \rangle + 4\langle \xi_{\xi_{e_i} \cdot}, e_i \rangle) \\
& + \frac{2(n-1)}{n} \sum_{A=I,J,K} \pi_{S^2 T^*}(\langle \cdot, \xi_{\xi_{e_i} A e_i} A \cdot \rangle + \langle \cdot, (\tilde{\nabla}_{e_i} \xi)_{A e_i} A \cdot \rangle) \\
& + \sum_{A=I,J,K} \pi_{\Lambda_0^2 E}(\langle \xi \cdot A, \xi_{e_i} A e_i \rangle + \langle \xi \cdot e_i, \xi_{A e_i} A \cdot \rangle) \\
& + \sum_{A=I,J,K} \pi_{\Lambda_0^2 E}(\frac{2}{n} \langle \xi \cdot e_i, \xi_A A e_i \rangle + \frac{2(n-1)}{n} \langle \xi_{e_i} \cdot, \xi_{A e_i} A \cdot \rangle) \\
& + 4\pi_{\Lambda_0^2 E}(\langle (\tilde{\nabla} \cdot \xi)_{e_i} \cdot, e_i \rangle - \langle (\tilde{\nabla}_{e_i} \xi) \cdot, e_i \rangle).
\end{aligned} \tag{6.28}$$

Similarly, using Proposition 4.5(vi) and (vii), we see that the curvature components in $(S^2 E S^2 H)_x$, for $x = a, b$, are determined by

$$\begin{aligned}
\operatorname{Ric}_{(S^2 E S^2 H)_a} &= \frac{1}{4}(\operatorname{Ric}_{S^2 E S^2 H} + 3 \operatorname{Ric}_{S^2 E S^2 H}^q), \\
\operatorname{Ric}_{(S^2 E S^2 H)_b} &= \frac{3}{4}(\operatorname{Ric}_{S^2 E S^2 H} - \operatorname{Ric}_{S^2 E S^2 H}^q),
\end{aligned}$$

which can be given in terms of ξ using equations (6.6), (6.10) and (6.17).

Theorem 6.10. *Let M be an almost quaternion-Hermitian $4n$ -manifold, $4n \geq 8$, with minimal $\operatorname{Sp}(n) \operatorname{Sp}(1)$ -connection $\tilde{\nabla} = \nabla + \xi$.*

(i) *Using equations (6.6), (6.10), (6.17), (6.24), (6.25), (6.26), (6.27) and (6.28) each of the tensors $\sum_{A=I,J,K} \langle \gamma_A, \omega_A \rangle$, $\tilde{\nabla} \xi_{UF}$ and $\xi_{UF} \odot \xi_{VG}$ contributes to the components of R in $\mathbb{R}_a + \mathbb{R}_b$, $(\Lambda_0^2 E)_a + (\Lambda_0^2 E)_b$, $(S^2 E S^2 H)_a + (S^2 E S^2 H)_b$*

and $\Lambda_0^2 ES^2H$ if and only if there is a tick in the corresponding place in Table 6.2. An entry with two ticks indicates independent contributions to both summands. For the modules \mathbb{R}_x , $\checkmark_{q(a,b)}^{p(a,b)}$ indicates that the contribution is a positive multiple of $p(a,b)$ and orthogonal to $q(a,b)$ with $a = \pi_2 + 6\pi_1$, $b = \pi_2 - 6\pi_1$.

(ii) Taking the image $\pi^\perp(R) = \pi_2 \circ \pi_1(R)$ into account, where $\pi_1(R)$ is given by equation (6.7), each of the tensors $\check{\nabla}\xi_{UF}$ and $\xi_{UF} \odot \xi_{VG}$ contributes to the components of R in V^{22} , $\Lambda_0^4 E$, $V^{31}S^2H$, $V^{211}S^2H$, $V^{22}S^4H$, $\Lambda_0^2 ES^4H$ and S^4H if and only if there is a tick in the corresponding place in Table 6.3. \square

A number of examples of almost quaternion-Hermitian manifolds with various different types of intrinsic torsion are given in Cabrera & Swann [12].

Corollary 6.11. *On an almost quaternion-Hermitian manifold that is quaternionic, i.e., $\xi \in (\Lambda_0^3 E + K + E)H$, there is no curvature in $V^{22}S^4H$, $\Lambda_0^2 ES^4H$ or S^4H .* \square

Corollary 6.12. *If ξ lies in $E(S^3H + H)$, then there is no curvature in V^{22} , $\Lambda_0^4 E$, $V^{31}S^2H$, $V^{211}S^2H$ or $V^{22}S^4H$.* \square

Corollary 6.13. *For $\xi \in \Lambda_0^3 E(S^3H + H)$ there is no curvature in $V^{31}S^2H$.* \square

Corollary 6.14. *Let $p \in M$. If ξ_p lies in $(\Lambda_0^3 E + E)S^3H + (\Lambda_0^3 E + K)H$ or in KS^3H and $\pi_{\mathbb{R}_a + \mathbb{R}_b}(R)$ is proportional to $2\pi_1 + \pi_2$ at p , then $\xi_p = 0$.* \square

The above result is a pointwise version of the following global theorem for compact manifolds.

Corollary 6.15 (Bor & Hernández Lamonedá [4]). *Suppose M is compact, that $\xi \in (\Lambda_0^3 E + E)S^3H + (\Lambda_0^3 E + K + E)H$ and that*

$$(n+2) \int_M \text{scal}^q \geq 3n \int_M \text{scal},$$

then $\xi = 0$ and M is quaternionic Kähler.

Proof. Subtracting the right-hand side from the left, the resulting integrand is a sum of the form $r_1 \|\xi_{33}\|^2 + r_2 \|\xi_{E3}\|^2 + r_3 \|\xi_{3H}\|^2 + r_4 \|\xi_{KH}\|^2 + r_5 \|\xi_{EH}\|^2 + r_6 d^* \theta^\xi$, with $r_1, \dots, r_5 < 0$. \square

$4n \geq 8$	\mathbb{R}_x	$(\Lambda_0^2 E)_x$	$(S^2 E S^2 H)_x$	$\Lambda_0^2 E S^2 H$
$\sum_A \langle \gamma_A, \omega_A \rangle$	$\sqrt{\frac{2a+b}{k_1 a - k_2 b}}$			
$\tilde{\nabla} \xi_{33}$			✓	✓
$\tilde{\nabla} \xi_{K3}$			✓	✓
$\tilde{\nabla} \xi_{E3}$			✓	✓
$\tilde{\nabla} \xi_{3H}$		✓	✓	✓
$\tilde{\nabla} \xi_{KH}$		✓	✓	✓
$\tilde{\nabla} \xi_{EH}$	$\sqrt{\frac{2k_1 a - k_2 b}{a+b}}$	✓	✓	✓
$\xi_{33} \otimes \xi_{33}$	$\sqrt{\frac{5k_1 a - k_2 b}{2a+5b}}$	✓	✓	
$\xi_{K3} \otimes \xi_{K3}$	$\sqrt{\frac{k_1 a + k_2 b}{2a-b}}$	✓✓	✓✓	
$\xi_{E3} \otimes \xi_{E3}$	$\sqrt{\frac{2k_1 f(n)a - k_2 g(n)b}{g(n)a + f(n)b}}$	✓	✓	
$\xi_{3H} \otimes \xi_{3H}$	$\sqrt{\frac{-14k_1 a - 5k_2 b}{5a-7b}}$	✓	✓	
$\xi_{KH} \otimes \xi_{KH}$	$\sqrt{\frac{-17k_1 a - 5k_2 b}{10a-7b}}$	✓✓	✓✓	
$\xi_{EH} \otimes \xi_{EH}$	$\sqrt{\frac{-h(n)a + k_2^2 b}{2k_1 k_2 a + h(n)b}}$	✓	✓	
$\xi_{33} \odot \xi_{K3}$		✓	✓	✓
$\xi_{33} \odot \xi_{E3}$		✓		✓
$\xi_{33} \odot \xi_{3H}$			✓	✓
$\xi_{33} \odot \xi_{KH}$			✓	✓
$\xi_{33} \odot \xi_{EH}$				✓
$\xi_{K3} \odot \xi_{E3}$			✓	✓
$\xi_{K3} \odot \xi_{3H}$			✓	✓
$\xi_{K3} \odot \xi_{KH}$			✓✓	✓
$\xi_{K3} \odot \xi_{EH}$			✓	✓
$\xi_{E3} \odot \xi_{3H}$				✓
$\xi_{E3} \odot \xi_{KH}$			✓	✓
$\xi_{E3} \odot \xi_{EH}$			✓	✓
$\xi_{3H} \odot \xi_{KH}$		✓	✓	✓
$\xi_{3H} \odot \xi_{EH}$		✓		✓
$\xi_{KH} \odot \xi_{EH}$		✓		✓

Table 6.2: Curvature complementary (I) to $S^4 E$, $4n \geq 8$, from Theorem 6.10(i). Here $x = a, b$, $k_1 = n - 1$, $k_2 = 2n + 1$, $f(n) = n^2 + 3n + 1$, $g(n) = n^2 + 1$, $h(n) = (2n - 1)(n + 1)$.

$4n \geq 8$	V^{22}	$\Lambda_0^4 E$	$V^{31} S^2 H$	$V^{211} S^2 H$	$V^{22} S^4 H$	$\Lambda_0^2 E S^4 H$	$S^4 H$
$\tilde{\nabla} \xi_{33}$				✓		✓	
$\tilde{\nabla} \xi_{K3}$			✓	✓	✓	✓	
$\tilde{\nabla} \xi_{E3}$						✓	✓
$\tilde{\nabla} \xi_{3H}$		✓		✓			
$\tilde{\nabla} \xi_{KH}$	✓		✓	✓			
$\tilde{\nabla} \xi_{EH}$							
$\xi_{33} \otimes \xi_{33}$	✓	✓		✓	✓	✓	✓
$\xi_{K3} \otimes \xi_{K3}$	✓	✓	✓		✓	✓	✓
$\xi_{E3} \otimes \xi_{E3}$						✓	✓
$\xi_{3H} \otimes \xi_{3H}$	✓	✓		✓			
$\xi_{KH} \otimes \xi_{KH}$	✓	✓	✓				
$\xi_{EH} \otimes \xi_{EH}$							
$\xi_{33} \odot \xi_{K3}$	✓	✓	✓	✓	✓	✓	
$\xi_{33} \odot \xi_{E3}$		✓		✓		✓	
$\xi_{33} \odot \xi_{3H}$				✓	✓	✓	✓
$\xi_{33} \odot \xi_{KH}$			✓	✓	✓	✓	
$\xi_{33} \odot \xi_{EH}$				✓		✓	
$\xi_{K3} \odot \xi_{E3}$	✓		✓	✓	✓	✓	
$\xi_{K3} \odot \xi_{3H}$			✓	✓	✓	✓	
$\xi_{K3} \odot \xi_{KH}$			✓	✓	✓	✓	✓
$\xi_{K3} \odot \xi_{EH}$			✓	✓	✓	✓	
$\xi_{E3} \odot \xi_{3H}$				✓		✓	
$\xi_{E3} \odot \xi_{KH}$			✓	✓	✓	✓	
$\xi_{E3} \odot \xi_{EH}$						✓	✓
$\xi_{3H} \odot \xi_{KH}$	✓	✓	✓	✓			
$\xi_{3H} \odot \xi_{EH}$		✓		✓			
$\xi_{KH} \odot \xi_{EH}$	✓		✓	✓			

Table 6.3: Curvature complementary (II) to $S^4 E$, $4n \geq 8$, from Theorem 6.10(ii).

A similar result was found by Ivanov & Minchev [10] in the special case of quaternionic Kähler manifolds with torsion, i.e., for $\xi \in (K + E)H$.

Corollary 6.16. *If $\xi \in (\Lambda_0^3 E + K + E)S^3 H$, then the components of the curvature in $(\Lambda_0^2 E)_a$, $(\Lambda_0^2 E)_b$, V^{22} and $\Lambda_0^4 E$ are determined by ξ tensorially.* \square

Remark 6.17. It is necessary to say which formulæ we use to derive the entries in Tables 6.1, 6.2 and 6.3 since there are non-trivial relations between the tensors $\tilde{\nabla}\xi_{UF}$ and $\xi_{UF} \odot \xi_{VG}$. These relations come from the Bianchi identity for the curvature R when expressed in terms of the curvature \tilde{R} of $\tilde{\nabla}$ and ξ . The modules affected are $\Lambda_0^2 ES^2 H$, $V^{211} S^2 H$ and $\Lambda_0^2 ES^4 H$, there are two such relations for $\Lambda_0^2 ES^2 H$ and one for each of the other two modules. This means that one can remove two ticks or one tick, respectively, from the corresponding column at the cost of introducing ticks elsewhere in the same column. We expect to be able to derive these relations from the corresponding components of the equation $d^2\Omega = 0$.

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